LABORATOIRE D'INFORMATIQUE
DE NANTES ATLANTIQUE

UMR CNRS 6241
Université de Nantes
Ecole des Mines de Nantes

Learning probabilistic finite automata

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Pattern Analysis, Statistical Modelling and Computational Learning

dépasser les frontières

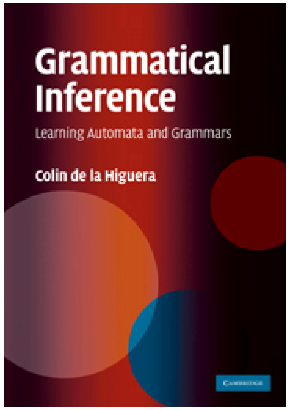
Région PAYS DE LA LOIRE

Acknowledgements

- Laurent Miclet, Jose Oncina, Tim Oates, Rafael Carrasco, Paco Casacuberta, Rémi Eyraud, Philippe Ezequel, Henning Fernau, Thierry Murgue, Franck Thollard, Enrique Vidal, Frédéric Tantini,...
- List is necessarily incomplete. Excuses to those that have been forgotten

http://pagesperso.lina.univ-nantes.fr/~cdlh/slides/

Chapters 5 and 16



Outline

1. PFA
2. Distances between distributions
3. FFA
4. Basic elements for learning PFA
5. ALERGIA
6. MDI and DSAI
7. Open questions

1 PFA

Probabilistic finite (state) automata

Practical motivations

(Computational biology, speech recognition, web services, automatic translation, image processing …)

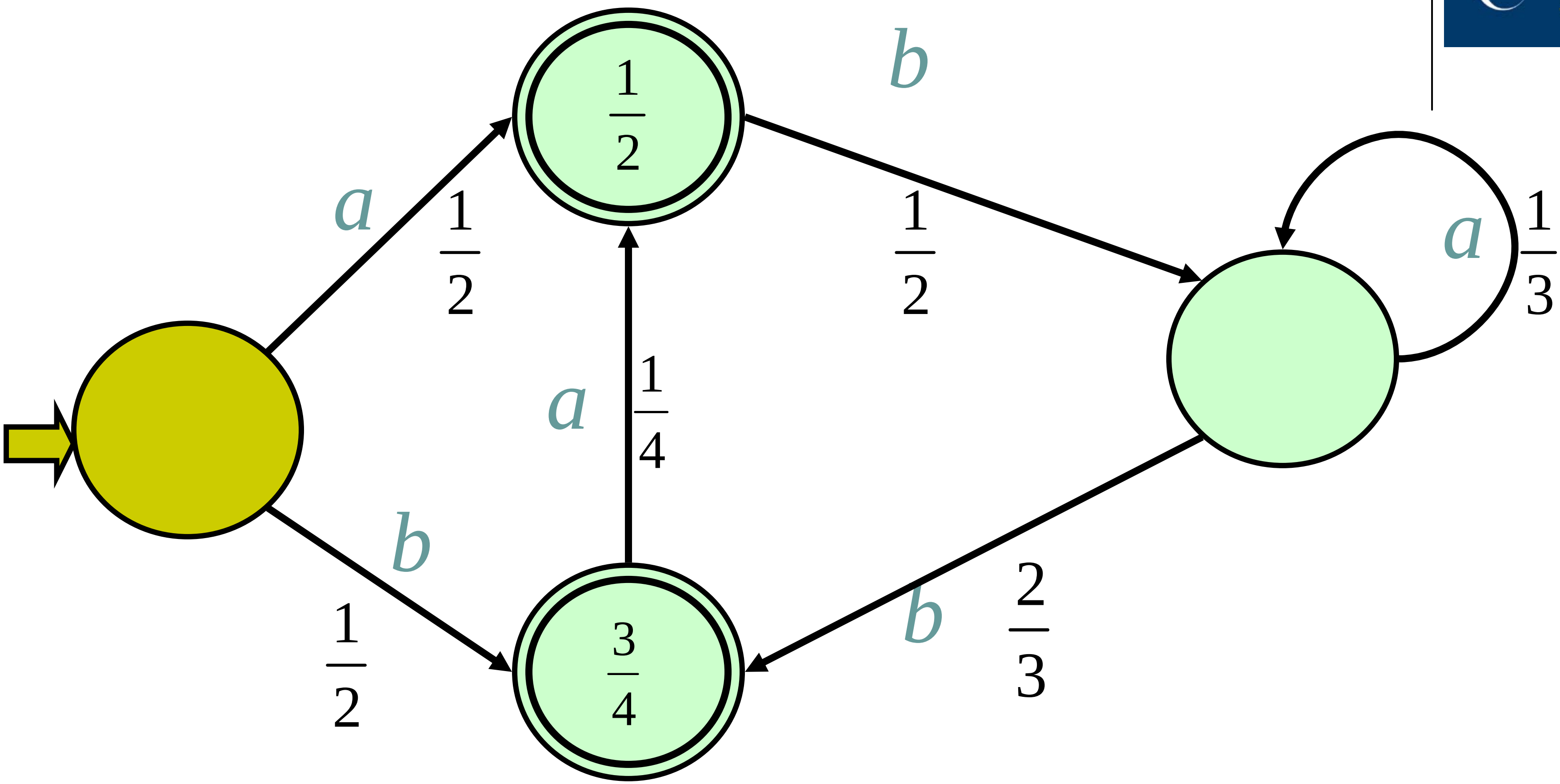
- A lot of positive data
- Not necessarily any negative data
- No ideal target
- Noise

The grammar induction problem, revisited

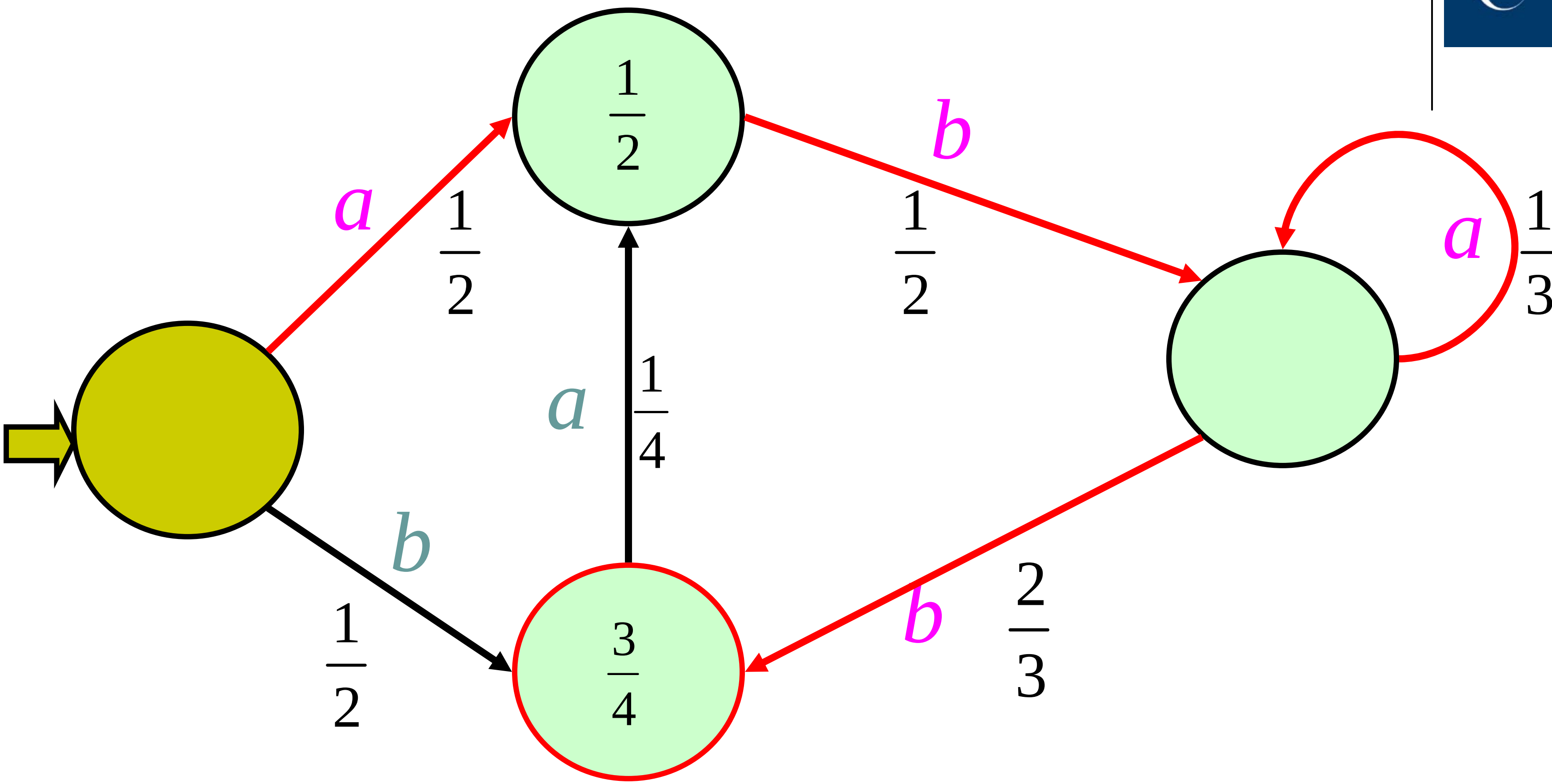
- The data consists of positive strings, «generated» following an unknown distribution
- The goal is now to find (learn) this distribution
- Or the grammar/automaton that is used to generate the strings

Success of the probabilistic models

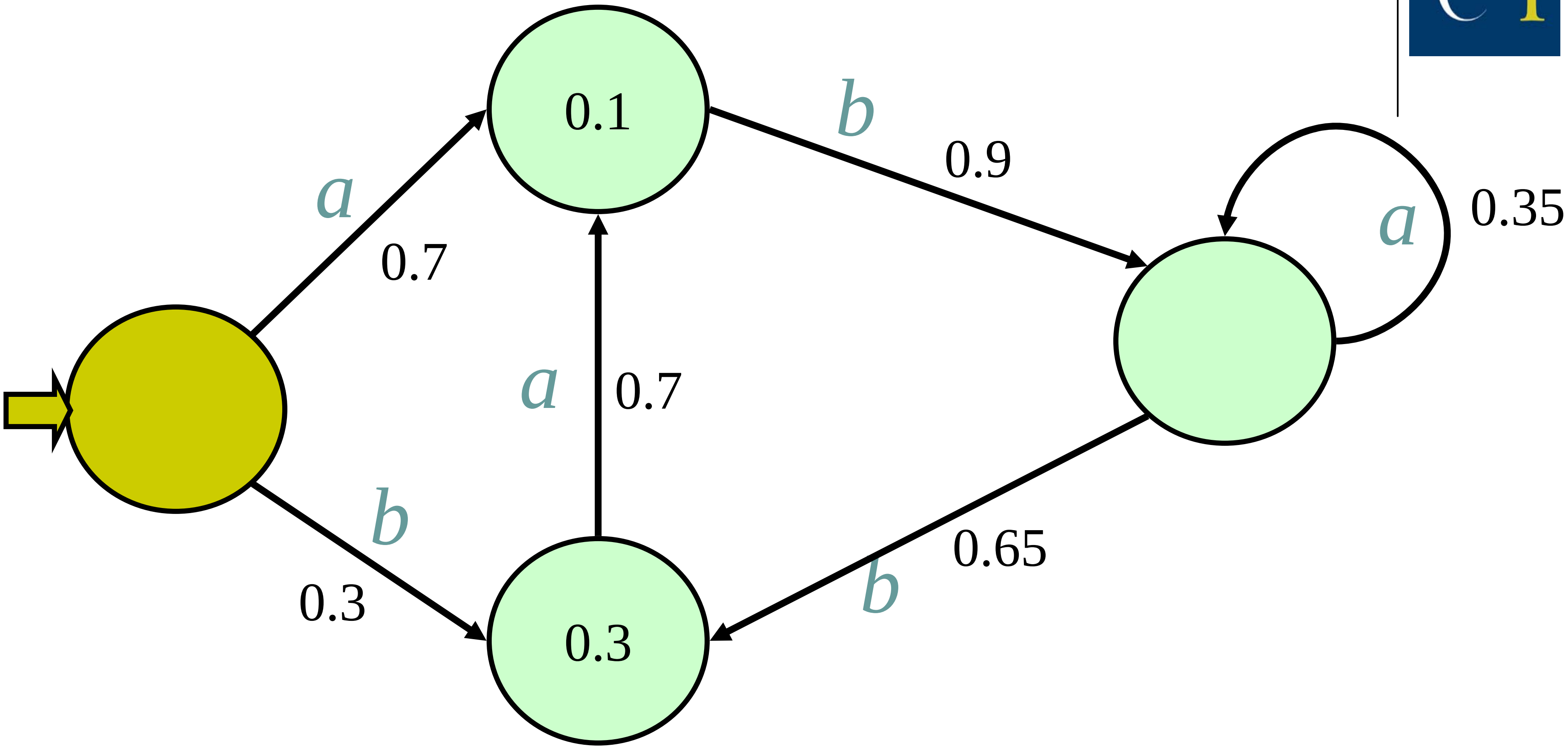
- *n*-grams
- Hidden Markov Models
- Probabilistic grammars



DPFA: Deterministic Probabilistic Finite Automaton



$$\Pr_A(abab) = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{3} \times \frac{2}{3} \times \frac{3}{4} = \frac{1}{24}$$


0.1
b
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b
0.65
b
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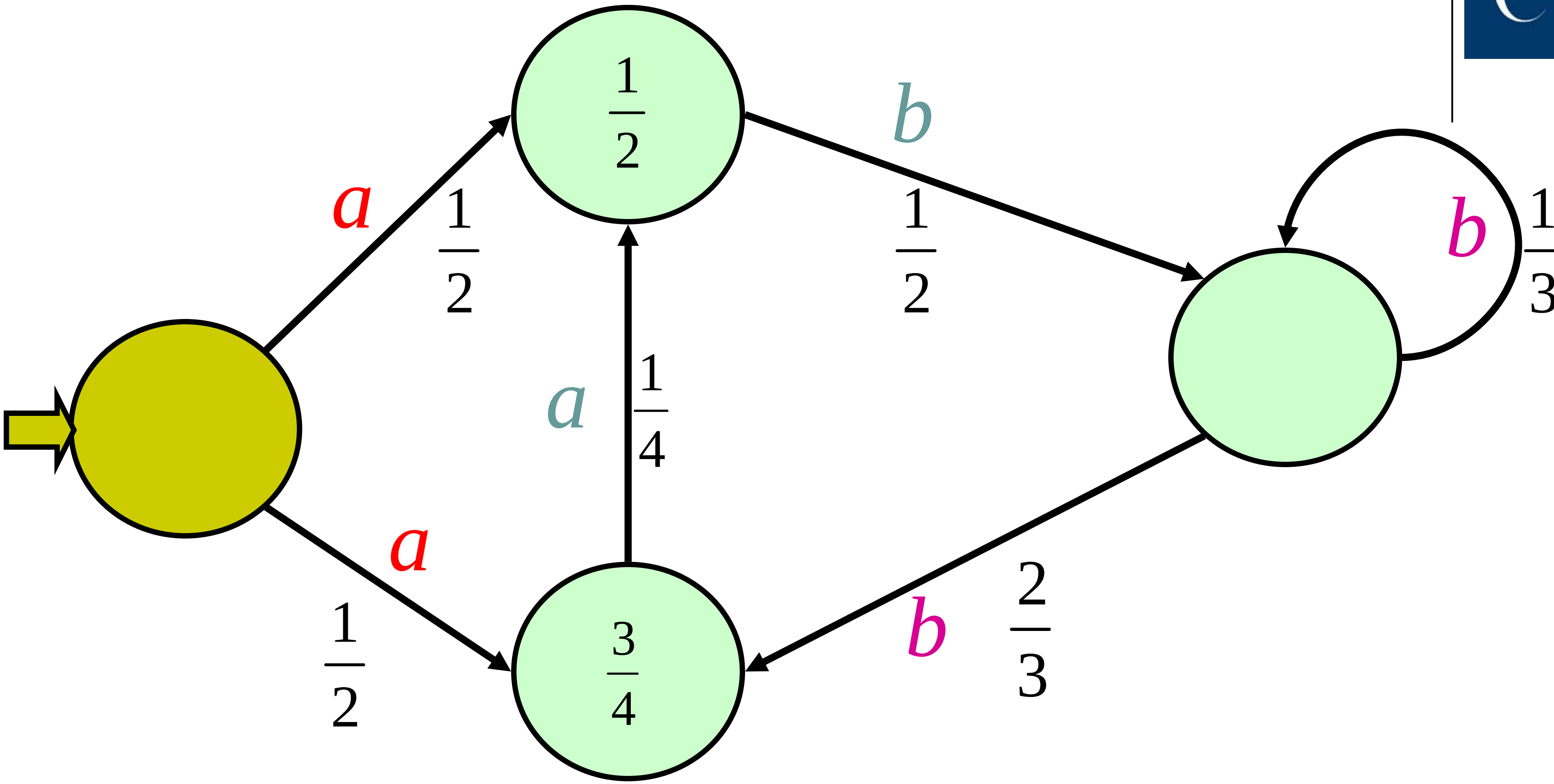
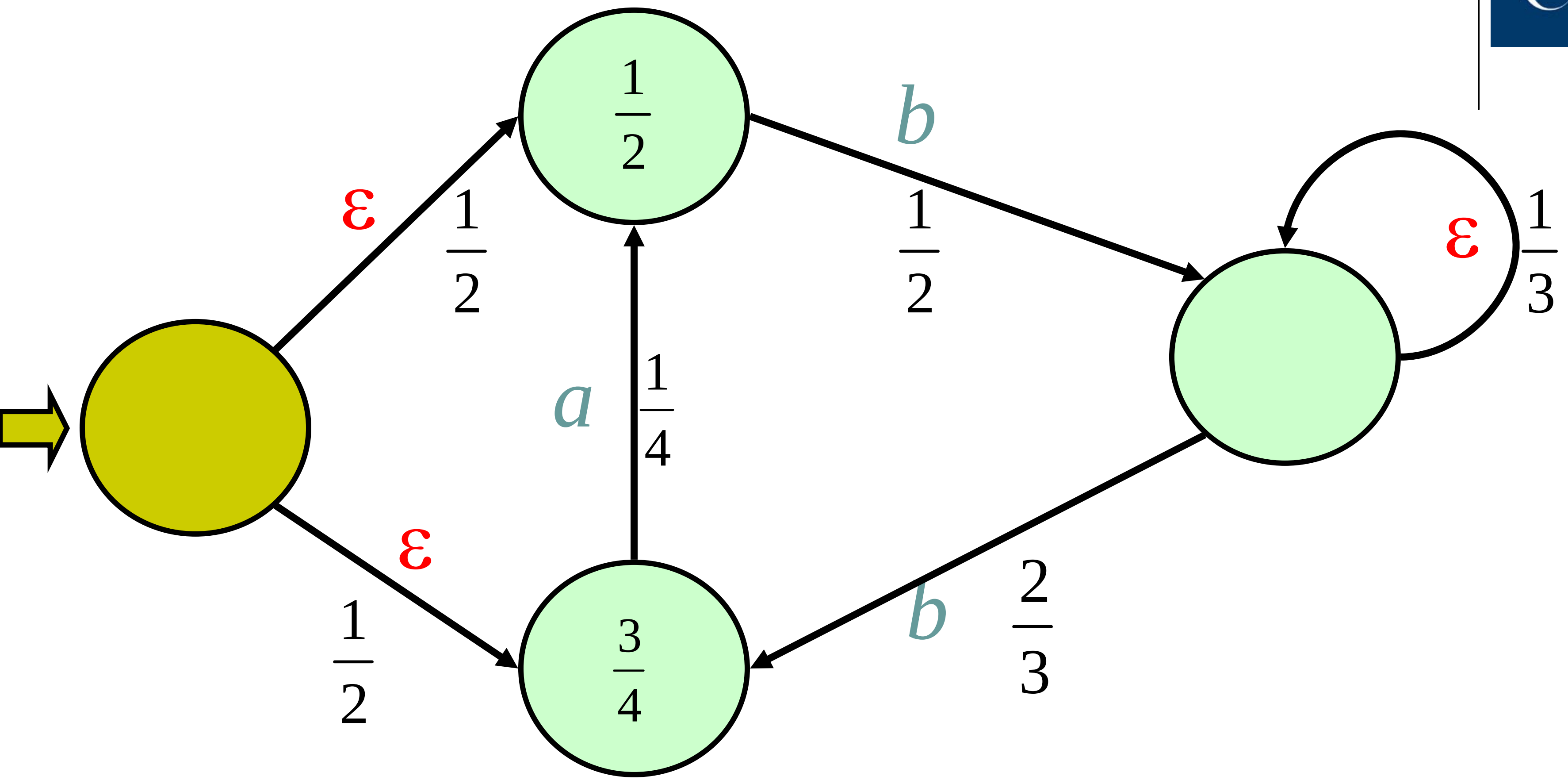
0.3



PFA: Probabilistic Finite (state) Automaton



ε-*PFA*: Probabilistic Finite (state) Automaton with ε-transitions

How useful are these automata?

- They can define a distribution over Σ^*
- They do not tell us if a string belongs to a language
- They are good candidates for grammar induction
- There is (was?) not that much written theory

Basic references

- The *HMM* literature
- Azaria Paz 1973: **Introduction to probabilistic automata**
- Chapter 5 of my book
- **Probabilistic Finite-State Machines**, Vidal, Thollard, cdlh, Casacuberta & Carrasco
- *Grammatical Inference* papers

Automata, definitions

Let $\mathcal{D}$ be a distribution over Σ^*

$$0 \leq Pr_{\mathcal{D}}(w) \leq 1$$

$$\sum_{w \in \Sigma^*} Pr_{\mathcal{D}}(w) = 1$$

A Probabilistic Finite (state) Automaton is a $\langle Q, \Sigma, I_P, F_P, \delta_P \rangle$

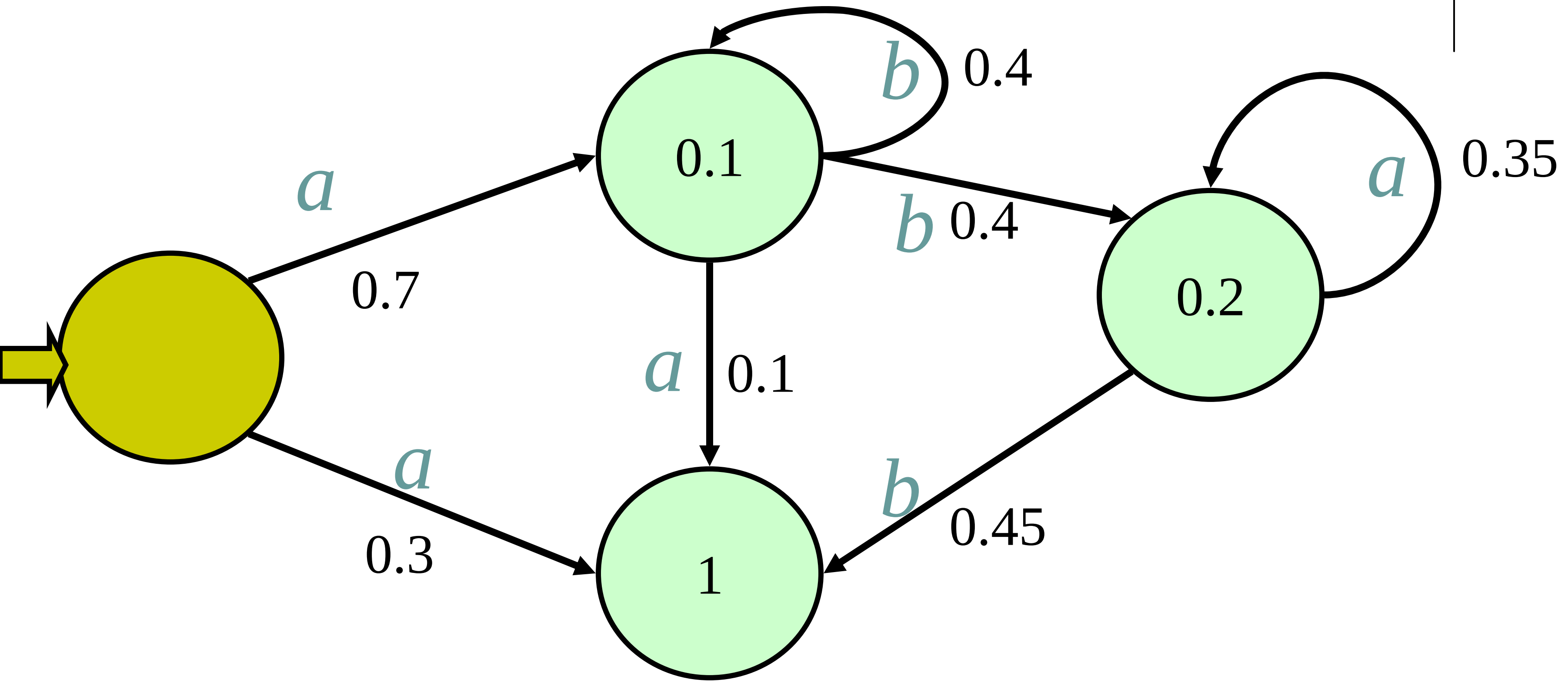
- Q set of states
- $I_P : Q \rightarrow [0;1]$
- $F_P : Q \rightarrow [0;1]$
- $\delta_P : Q \times \Sigma \times Q \rightarrow [0;1]$

What does a PFA do?

- It defines the probability of each string *w* as the sum (over all paths reading *w*) of the products of the probabilities

- $Pr_{\mathcal{A}}(w)=\sum_{\pi_i \in paths(w)} Pr(\pi_i)$
- $\pi_i=q_{i0}a_{i1}q_{i1}a_{i2}\ldots a_{in}q_{in}$
- $Pr(\pi_i)=I_P(q_{i0})\cdot F_P(q_{in}) \cdot \prod_{a_{ij}} \delta_P(q_{ij-1},a_{ij},q_{ij})$

- Note that if λ-transitions are allowed the sum may be infinite



Pr(*aba*) = 0.7*0.4*0.1*1 +0.7*0.4*0.45*0.2
= 0.028+0.0252=0.0532

- non deterministic *PFA*: many initial states/only one initial state
- an λ-*PFA*: a *PFA* with λ-transitions and perhaps many initial states
- *DPFA*: a deterministic *PFA*

Consistency

A PFA is consistent if

- $Pr_A(\Sigma^*)=1$
- $\forall x\in\Sigma^*\ 0\leq Pr_A(x)\leq 1$

Consistency theorem

A is consistent *if* every state is useful (accessible and co-accessible) and

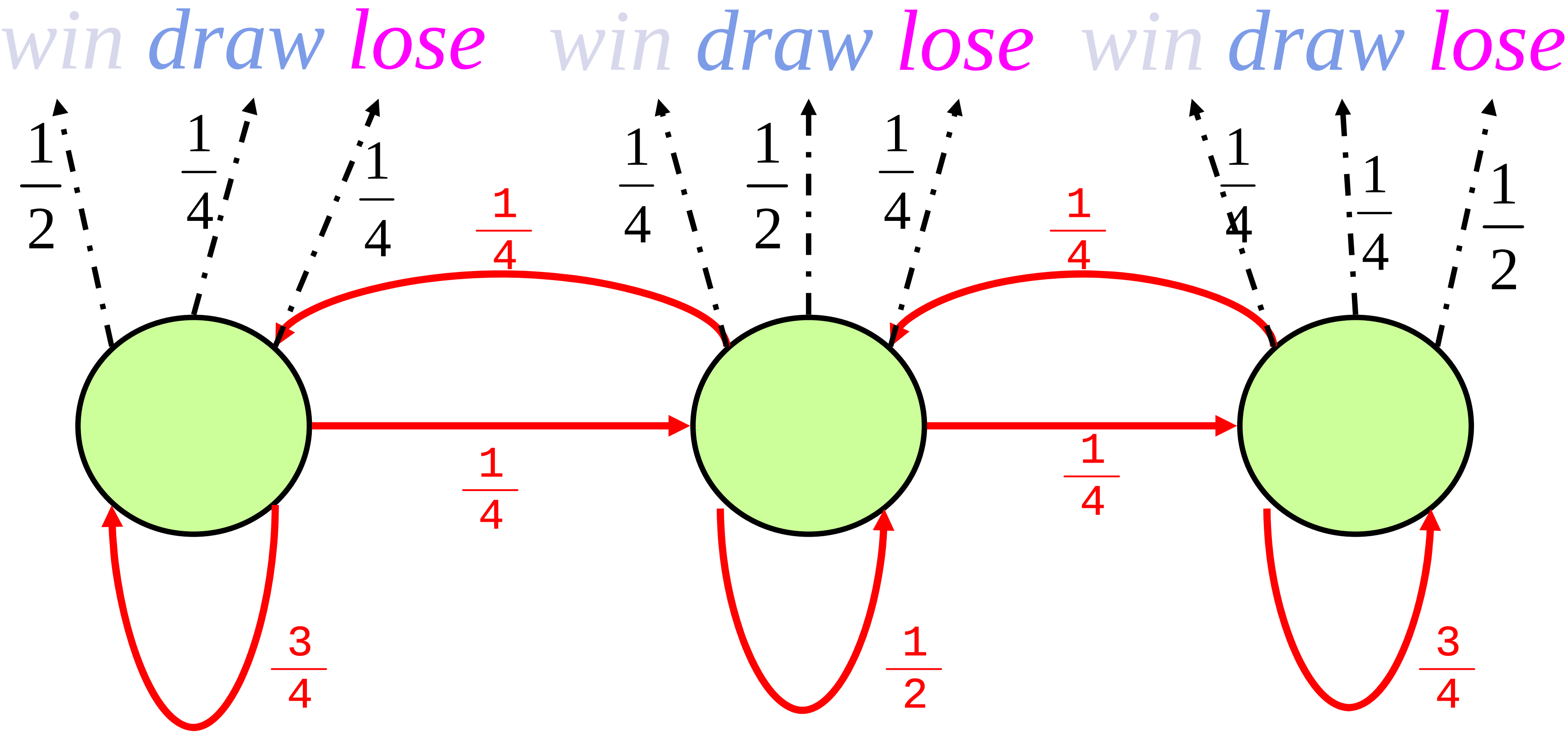
$$\forall q \in Q$$

$$F_P(q) + \sum_{q' \in Q, a \in \Sigma} \delta_P(q,a,q') = 1$$

Equivalence between models

- Equivalence between *PFA* and *HMM*…
- But the *HMM* usually define distributions over each Σ^n

A *football* HMM



Equivalence between *PFA* with λ-transitions and *PFA* without λ-transitions

cdlh 2003, Hanneforth & cdlh 2009

- Many initial states can be transformed into one initial state with λ-transitions;
- λ-transitions can be removed in polynomial time;
- Strategy:
  - number the states
  - eliminate first λ-loops, then the transitions with highest ranking arrival state

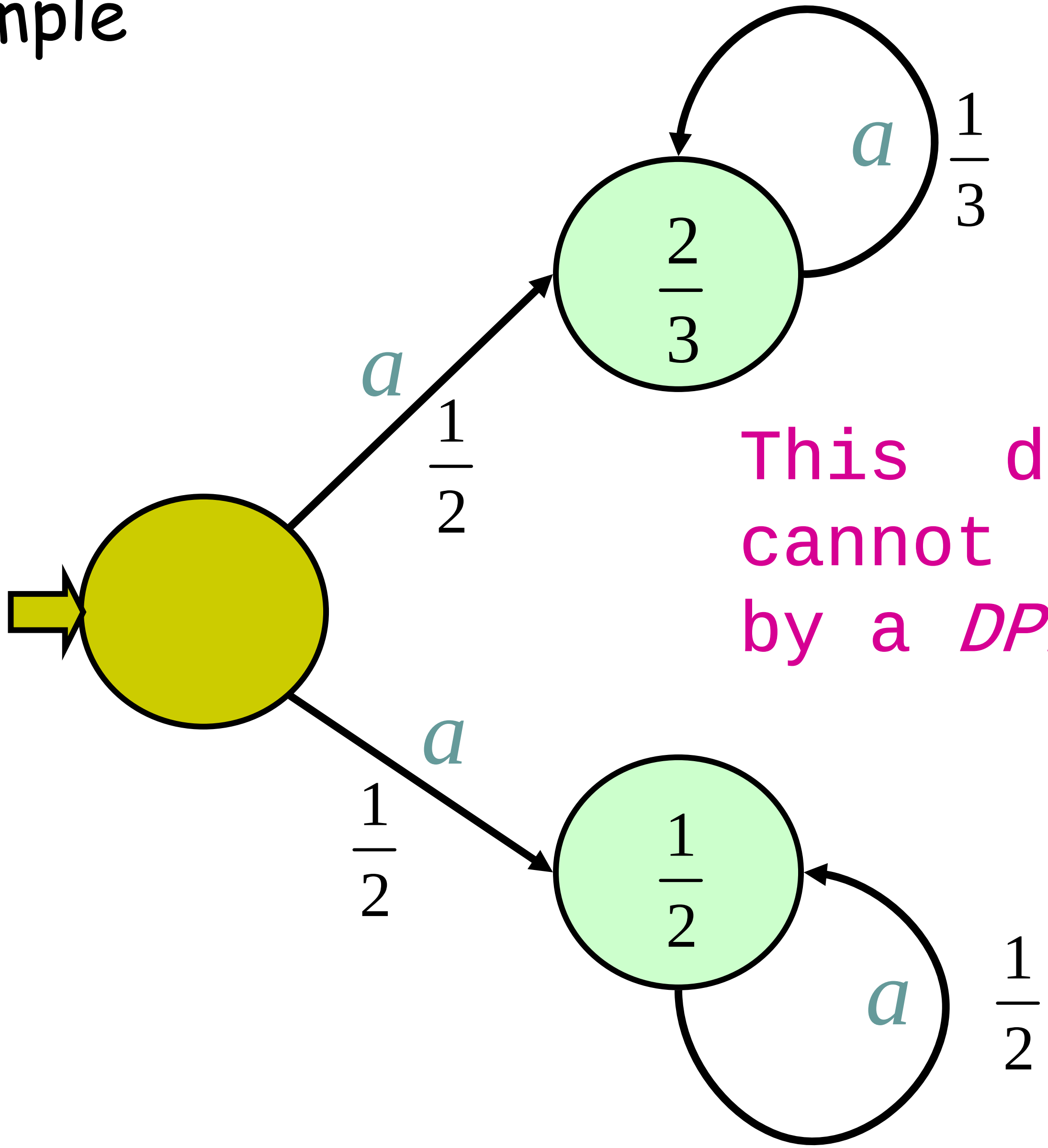
PFA are strictly more powerful than *DPFA*

Folk theorem

(and) You can't even tell in advance if you are in a good case or not

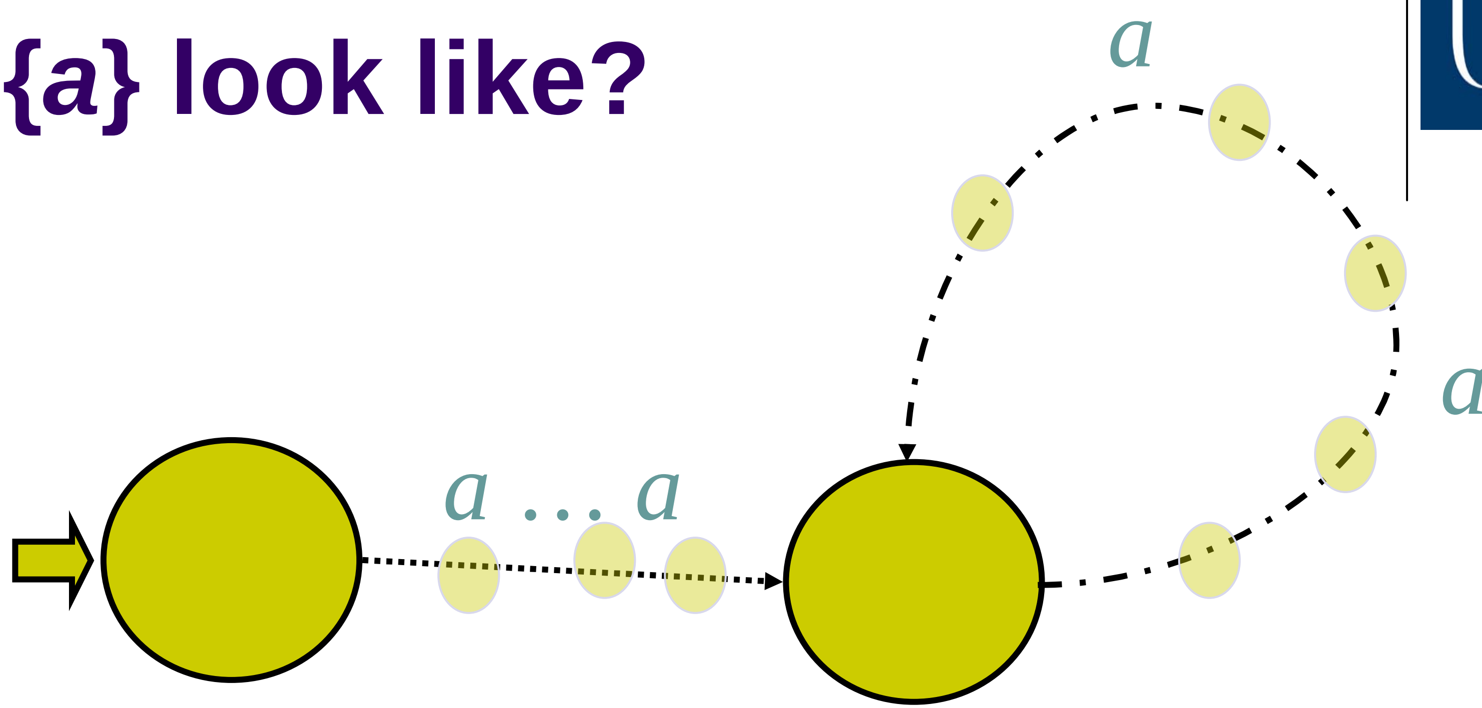
(see: Denis & Esposito 2004)

Example



This distribution cannot be modelled by a *DPFA*

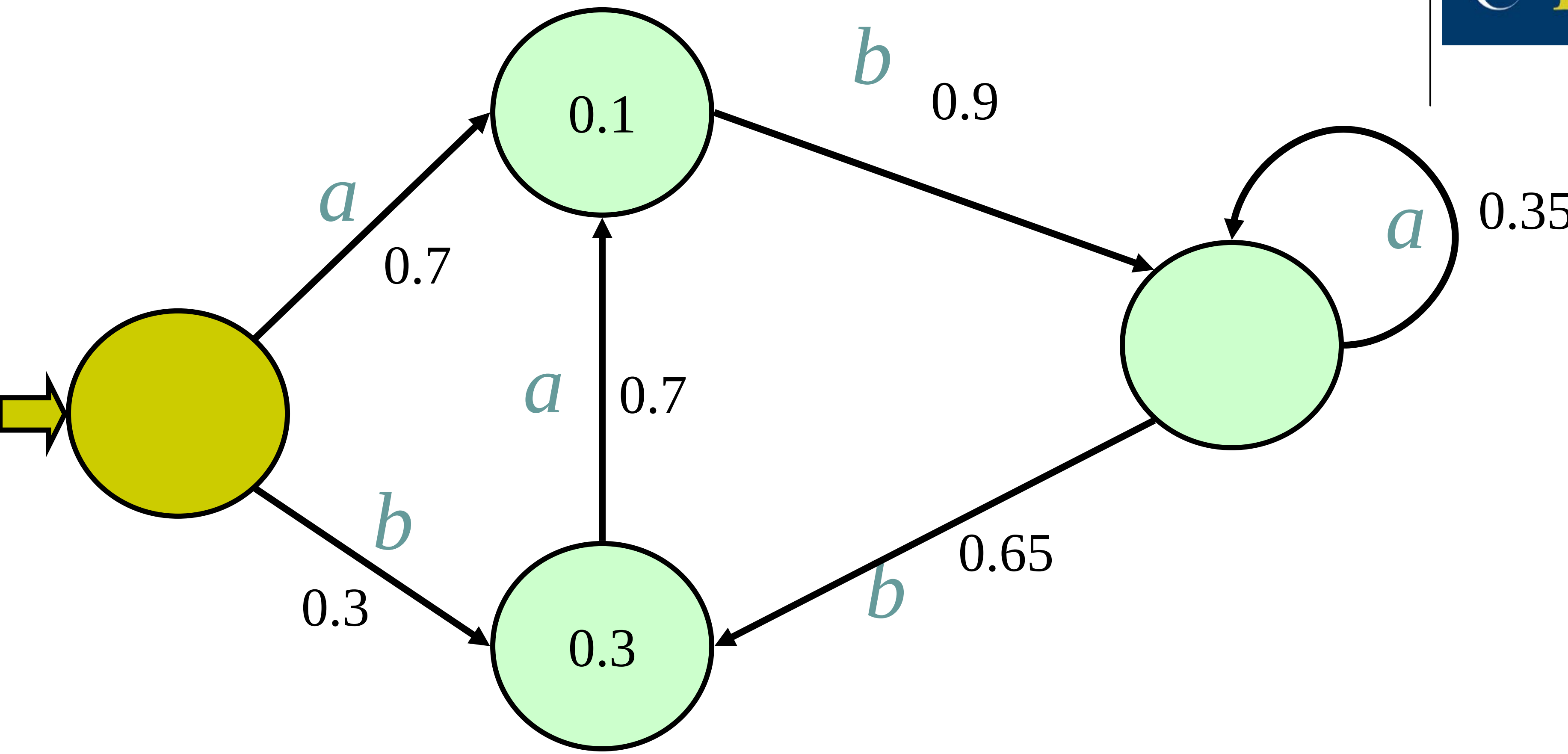
What does a *DPFA* over Σ ={*a*} look like?



And with this architecture you cannot generate the previous one

Parsing issues

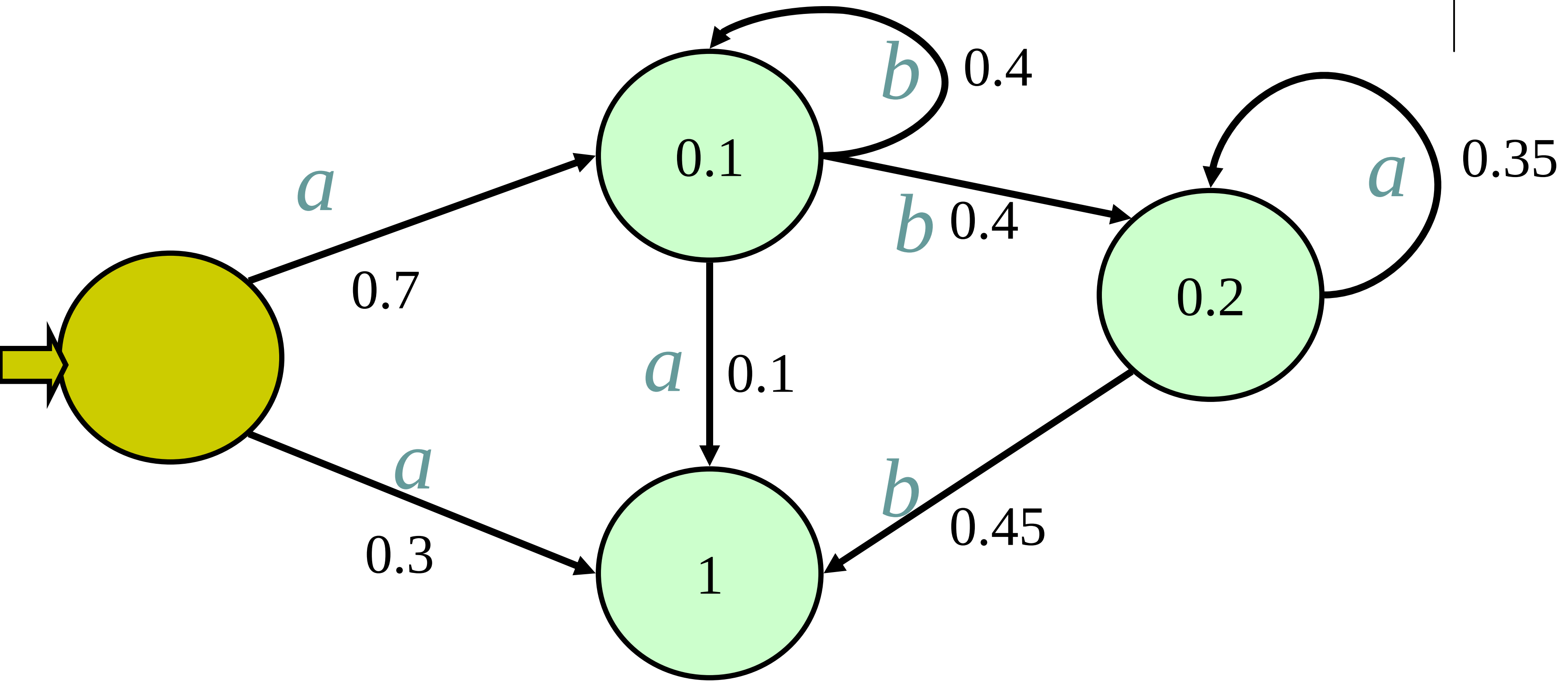
- Computation of the probability of a string or of a set of strings
- Deterministic case
  - Simple: apply definitions
  - Technically, rather sum up logs: this is easier, safer and cheaper



Pr(*aba*) = 0.7*0.9*0.35*0 = 0

Pr(*abb*) = 0.7*0.9*0.65*0.3 = 0.12285

Non-deterministic case



Pr(*aba*) = 0.7*0.4*0.1*1 +0.7*0.4*0.45*0.2
= 0.028+0.0252=0.0532

In the literature

- The computation of the probability of a string is by dynamic programming : $O(n^2 m)$
- 2 algorithms: *Backward* and *Forward*
- If we want the most probable derivation to define the probability of a string, then we can use the *Viterbi* algorithm

Forward algorithm

- $A[i,j]=\Pr(q_i | a_1..a_j)$
  (The probability of being in state q_i after having read $a_1..a_j$)
- $A[i,0]=I_P(q_i)$
- $A[i,j+1]= \sum_{k\leq|Q|} A[k,j] \cdot \delta_P(q_k,a_{j+1},q_i)$
- $\Pr(a_1..a_n)= \sum_{k\leq|Q|} A[k,n] \cdot F_P(q_k)$

2 Distances

What for?

- Estimate the quality of a language model
- Have an indicator of the convergence of learning algorithms
- Construct kernels

2.1 Entropy

- How many bits do we need to correct our model?
- Two distributions over Σ^*: $\mathcal{D}$ et $\mathcal{D}'$
- *Kullback Leibler divergence* (or relative entropy) between $\mathcal{D}$ and $\mathcal{D}'$:

$$\sum_{w \in \Sigma^*} Pr_{\mathcal{D}}(w) \times \left| \log Pr_{\mathcal{D}}(w) - \log Pr_{\mathcal{D}'}(w) \right|$$

2.2 Perplexity

- The idea is to allow the computation of the divergence, but relatively to a test set (*S*)
- An approximation (*sic*) is perplexity: inverse of the geometric mean of the probabilities of the elements of the test set

$$\prod_{w \in S} \Pr_{\mathcal{D}}(w)^{-1/|S|}$$

$$=$$

$$\frac{1}{\sqrt[|S|]{\prod_{w \in S} \Pr_{\mathcal{D}}(w)}}$$

Problem if some probability is null...

Why multiply (1)

- We are trying to compute the probability of independently drawing the different strings in set S

Why multiply? (2)

- Suppose we have two predictors for a coin toss
  - Predictor 1: heads 60%, tails 40%
  - Predictor 2: heads 100%
- The tests are H: 6, T: 4
- Arithmetic mean
  - P1: 36%+16%=0,52
  - P2: 0,6
- Predictor 2 is the better predictor ;-)

2.3 Distance d_2

$$d_2(\mathcal{D}, \mathcal{D}') = \sqrt{\sum_{w \in \Sigma^*} (\Pr_{\mathcal{D}}(w) - \Pr_{\mathcal{D}'}(w))^2}$$

Can be computed in polynomial time if $\mathcal{D}$ and $\mathcal{D}'$ are given by *PFA* (Carrasco & cdlh 2002)

This also means that equivalence of PFA is in **P**

3 FFA

Frequency Finite (state) Automata

A learning sample

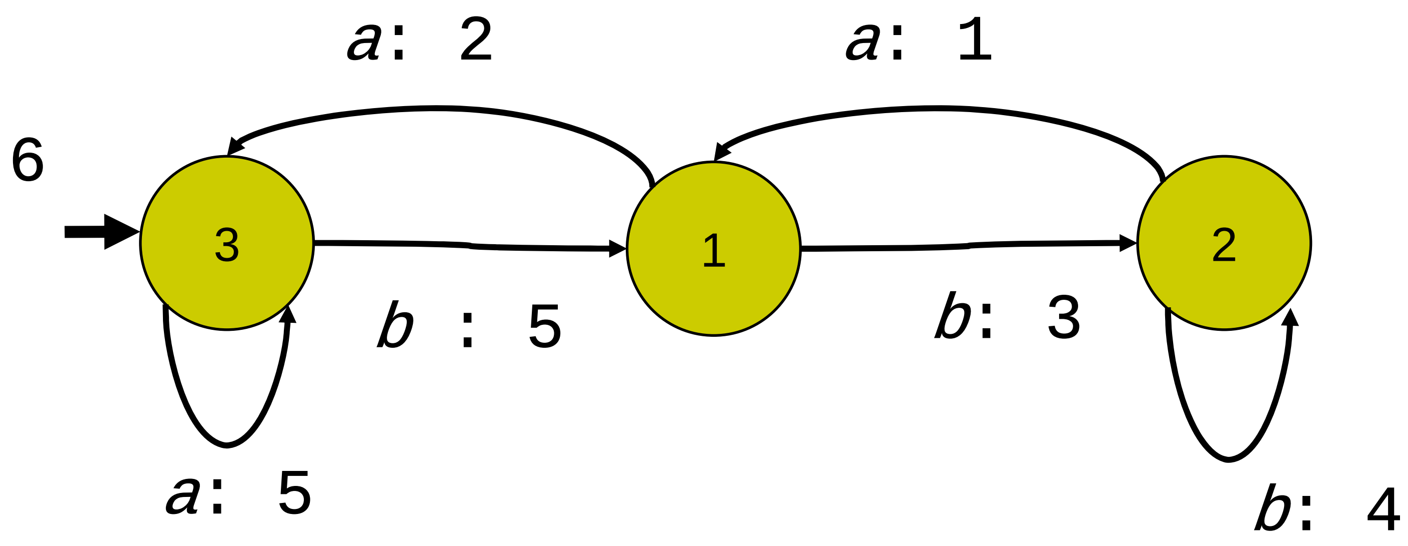
- is a multiset
- Strings appear with a frequency (or multiplicity)
- S={λ (3), *aaa* (4), *aaba* (2), *ababa* (1), *bb* (3), *bbaaa* (1)}

DFFA

A deterministic frequency finite automaton is a DFA with a frequency function returning a positive integer for every state and every transition, and for entering the initial state such that

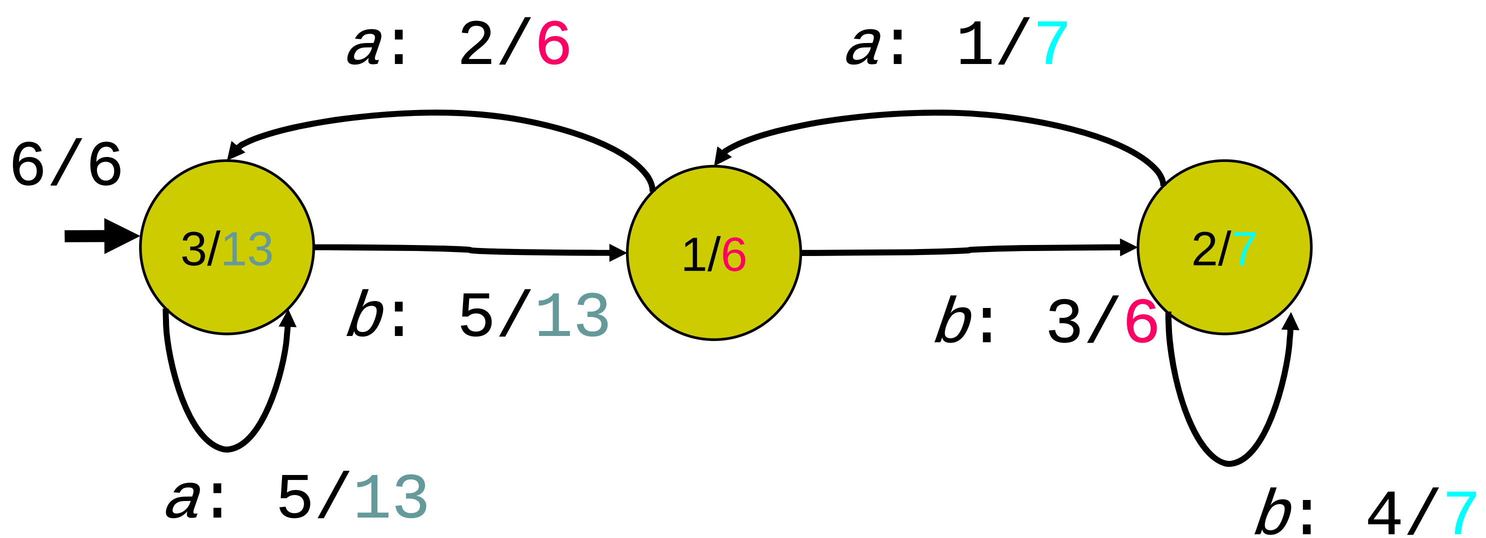
- the sum of what enters is equal to what exits and
- the sum of what halts is equal to what starts

Example



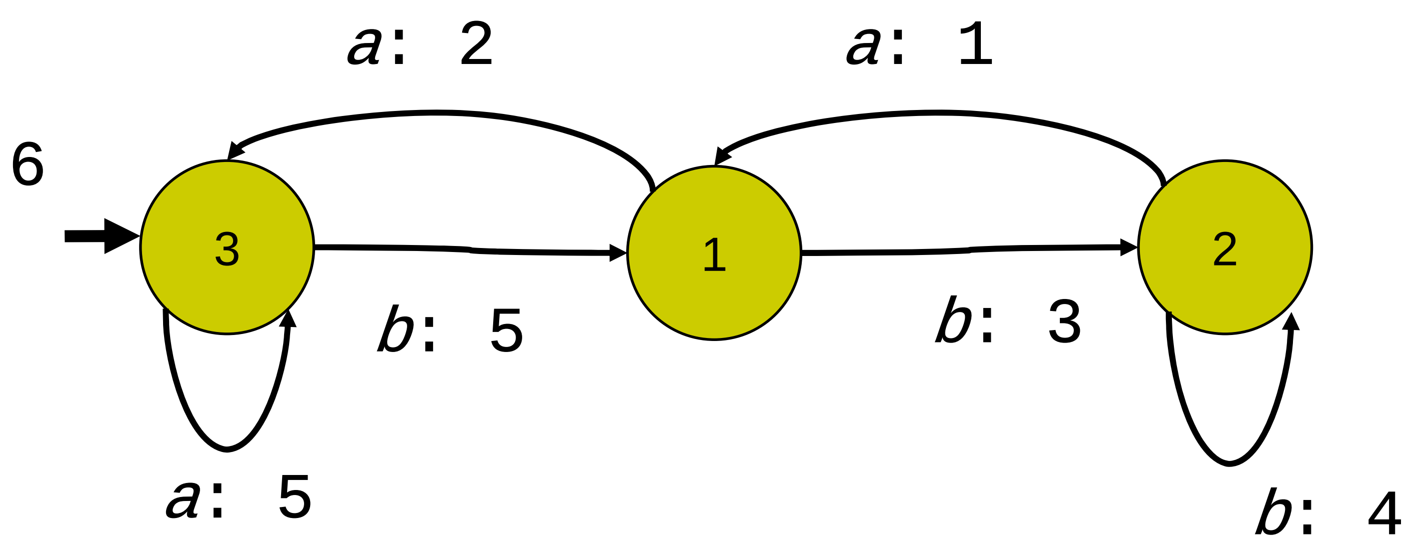
From a DFFA to a DPFA

Frequencies become relative frequencies by dividing by sum of exiting frequencies



From a DFA and a sample to a DFFA

S = {λ, *aaaa*, *ab*, *babb*, *bbbb*, *bbbbaa*}



Note

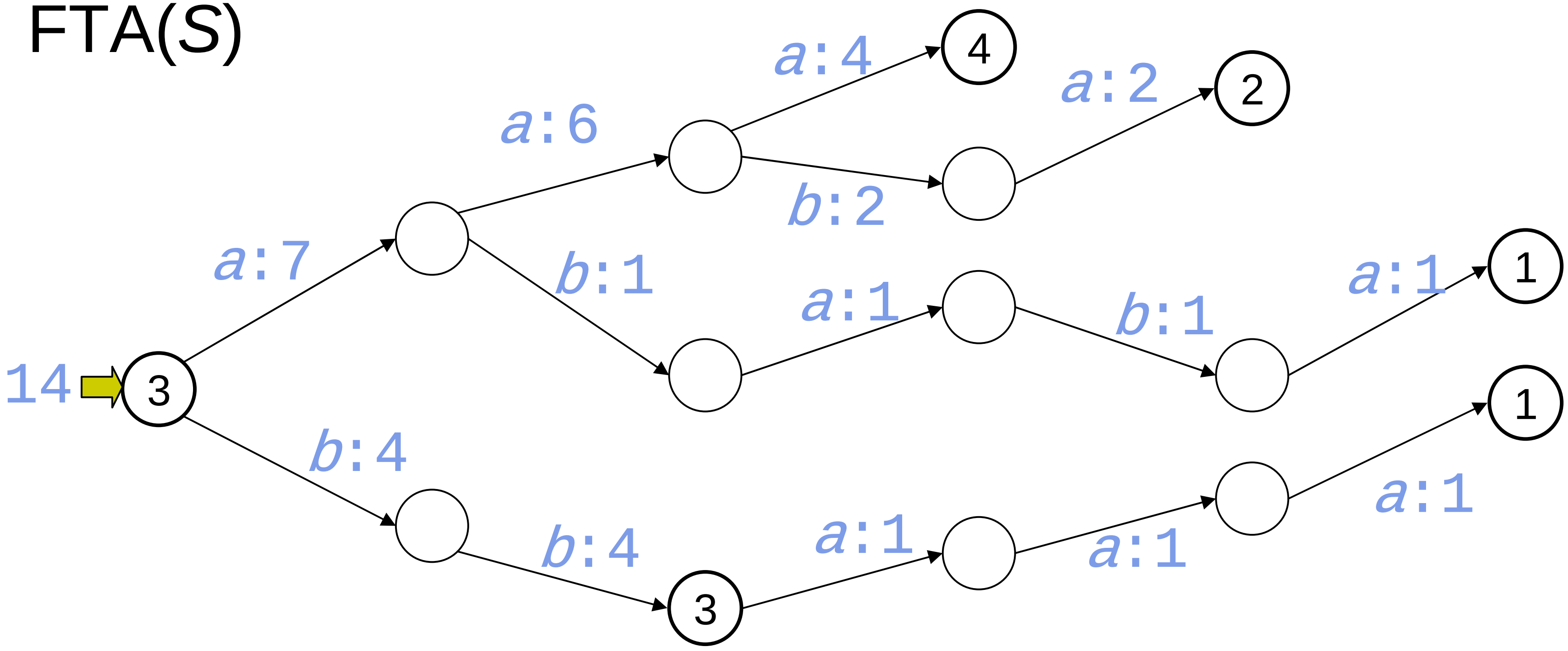
- Another sample may lead to the same DFFA
- Doing the same with a NFA is a much harder problem
- Typically what algorithm Baum-Welch (EM) has been invented for…

The frequency prefix tree acceptor

- The data is a multi-set
- The FTA is the smallest tree-like FFA consistent with the data
- Can be transformed into a PFA if needed

From the sample to the FTA

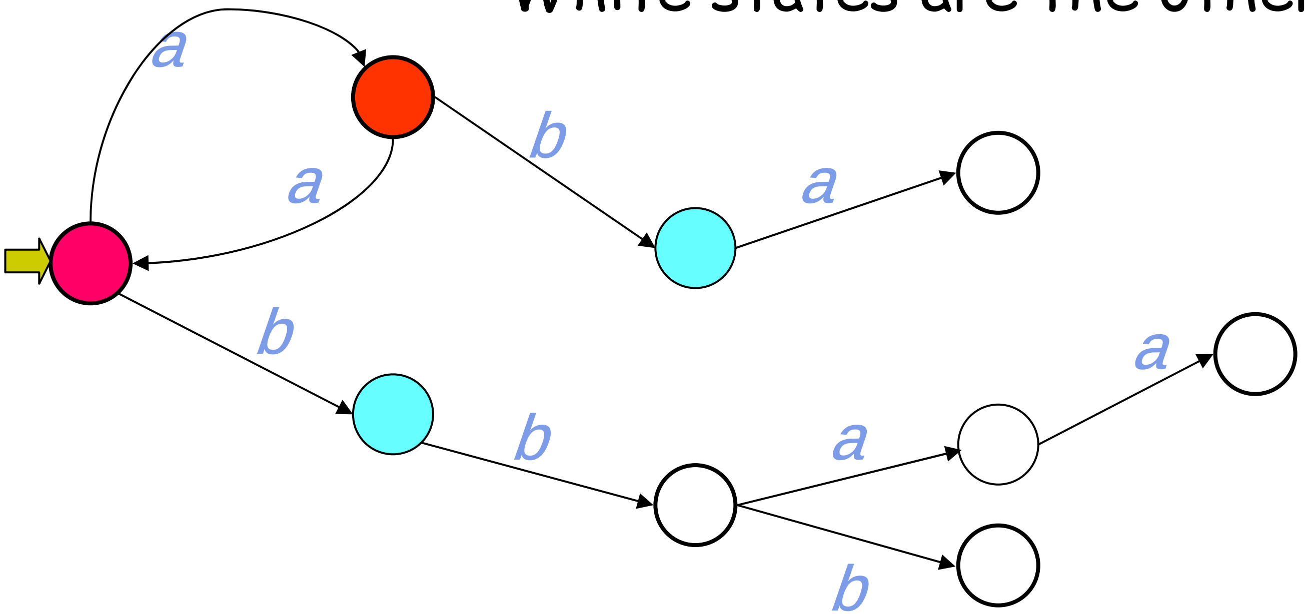
FTA(*S*)



S={λ (3), *aaa* (4), *aaba* (2), *ababa* (1), *bb* (3), *bbaaa* (1)}

Red, Blue and White states

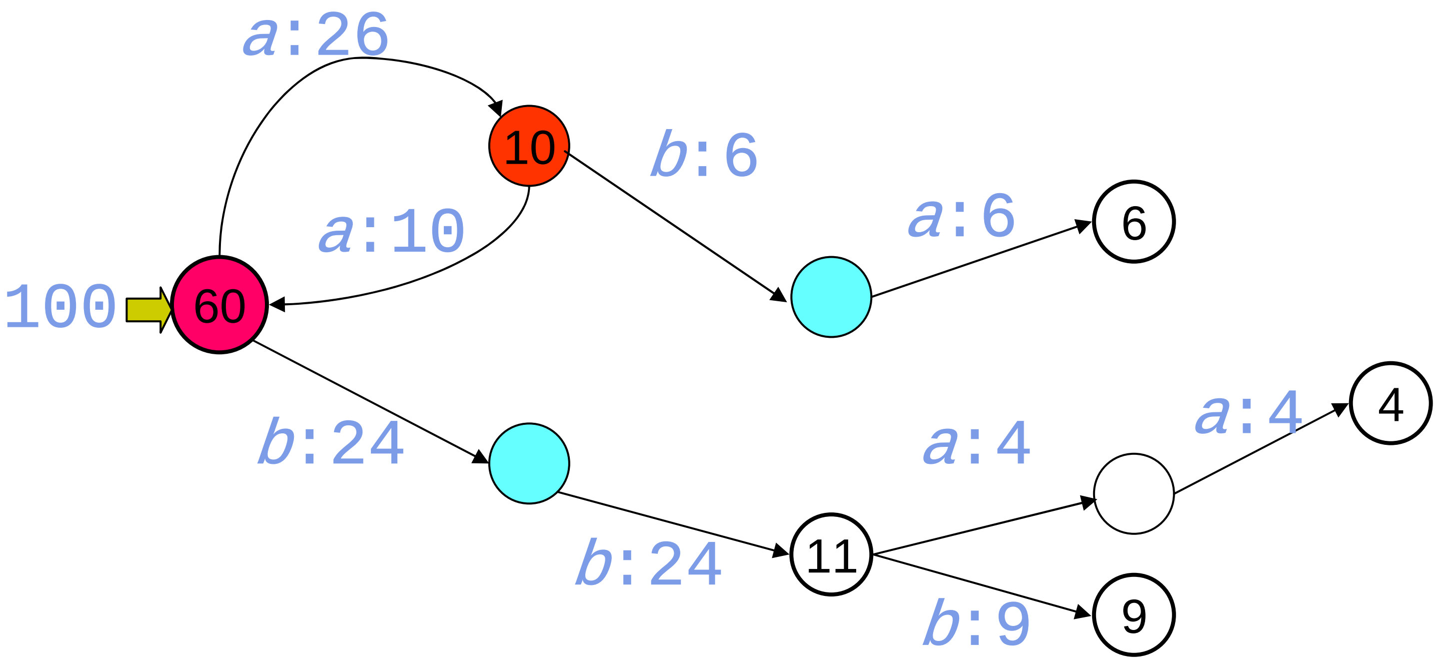
-Red states are confirmed states
-Blue states are the (non Red) successors of the Red states
-White states are the others



Same as with DFA and what RPNI does

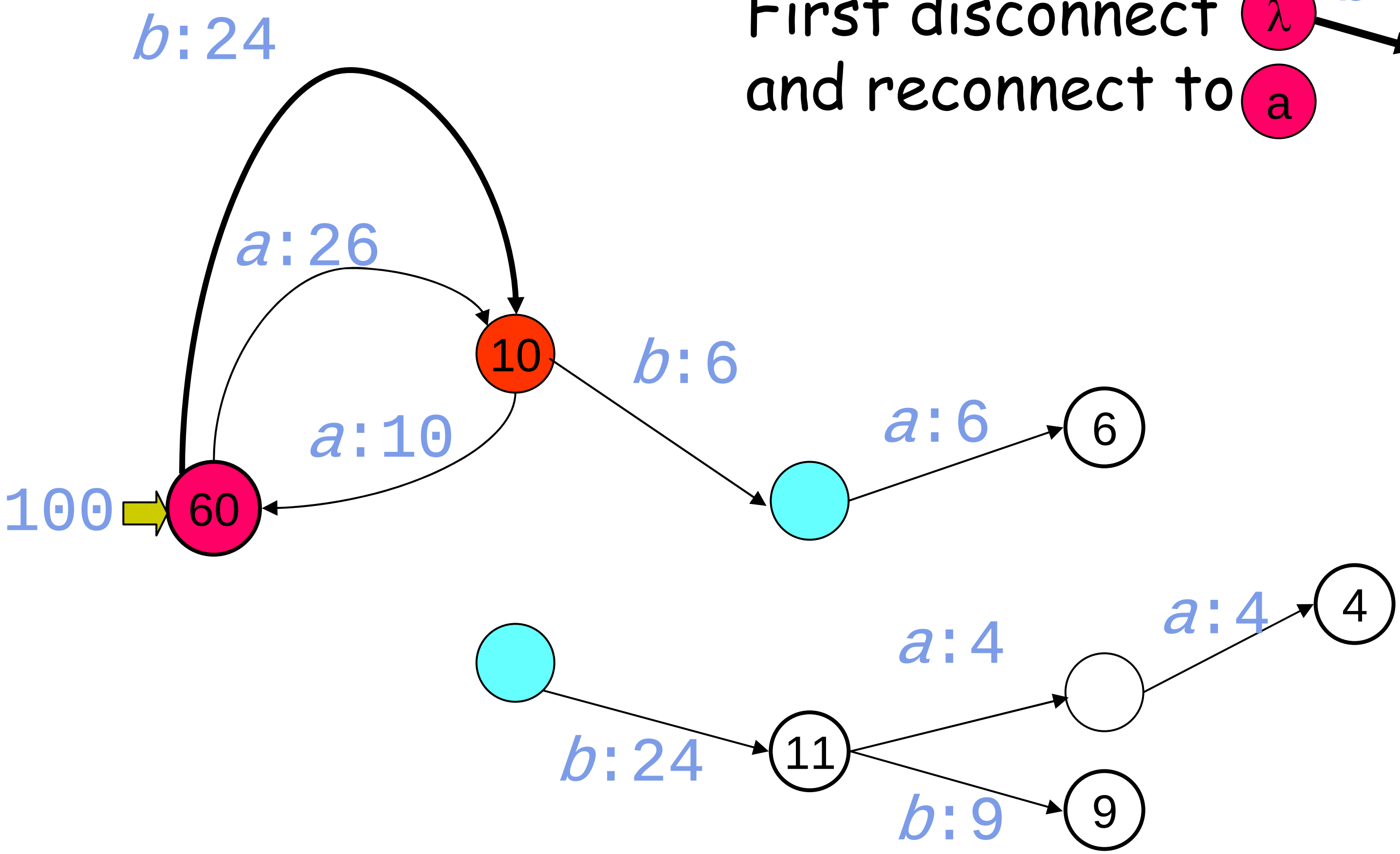
Merge and fold

Suppose we decide to merge λ —*b*→ with state a



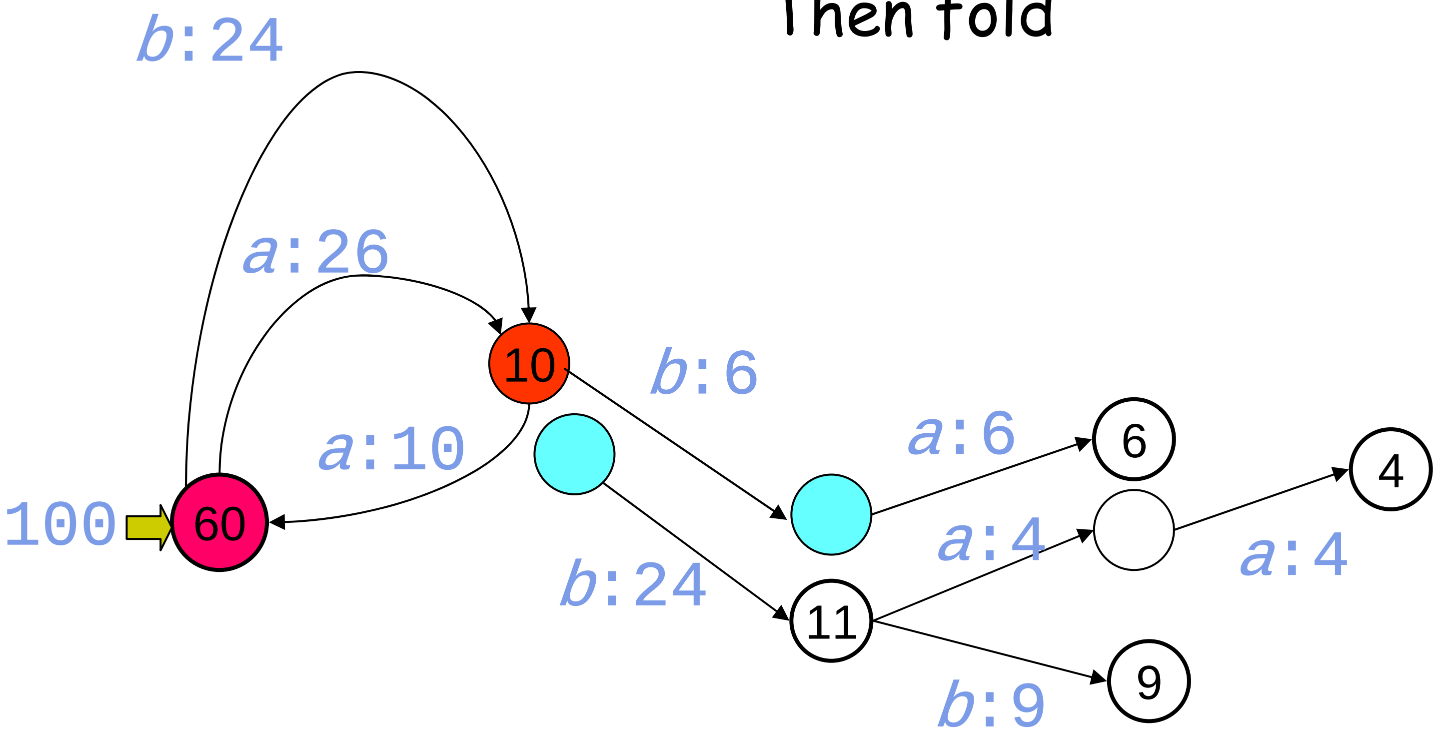
Merge and fold

First disconnect λ $\xrightarrow{b}$ and reconnect to a



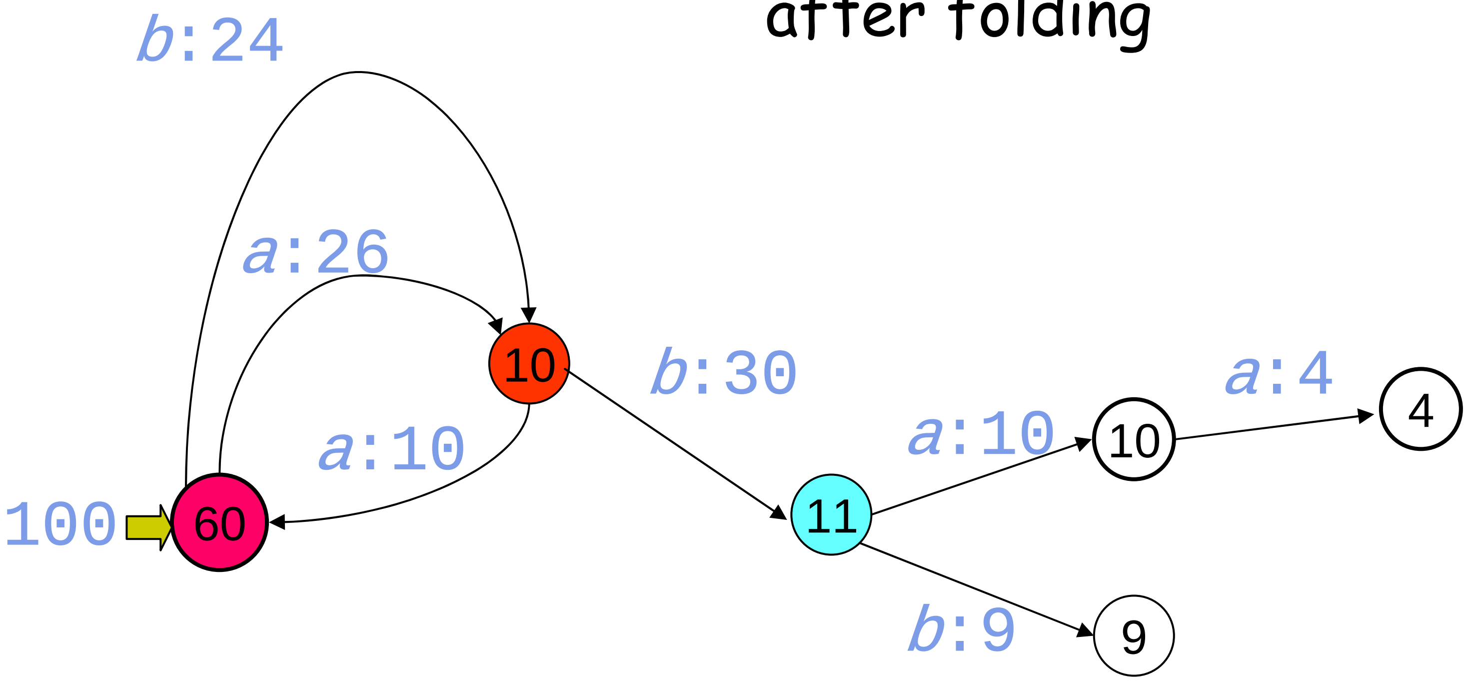
Merge and fold

Then fold



Merge and fold

after folding



State merging algorithm

A=FTA(S); *Blue* ={$\delta(q_I,a)$: $a\in\Sigma$ };

Red ={q_I}

While *Blue*$\neq\varnothing$ do

choose q from *Blue* such that Freq(q)$\geq t_0$

if $\exists p\in$ *Red*: d(A_p,A_q) is small

then A = merge_and_fold(A,p,q)

else *Red* = *Red* $\cup$ {q}

Blue = {$\delta(q,a)$: $q\in$ *Red*} – {*Red*}

The real question

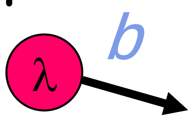
- How do we decide if d(A_p,A_q) is small?
- Use a distance…
- Be able to compute this distance
- If possible update the computation easily
- Have properties related to this distance

Deciding if two distributions are similar

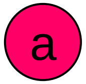
- If the two distributions are known, equality can be tested
- Distance (L_2 norm) between distributions can be exactly computed
- But what if the two distributions are unknown?

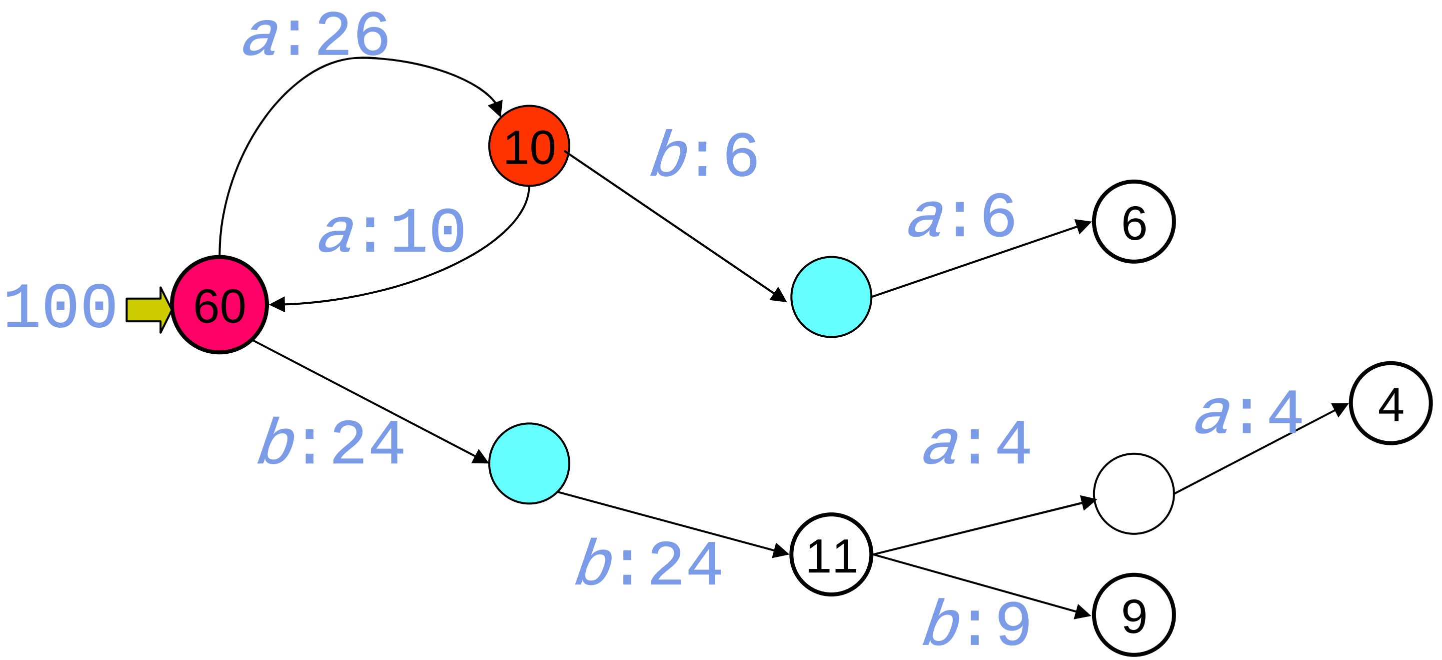
Taking decisions

Suppose we want to merge



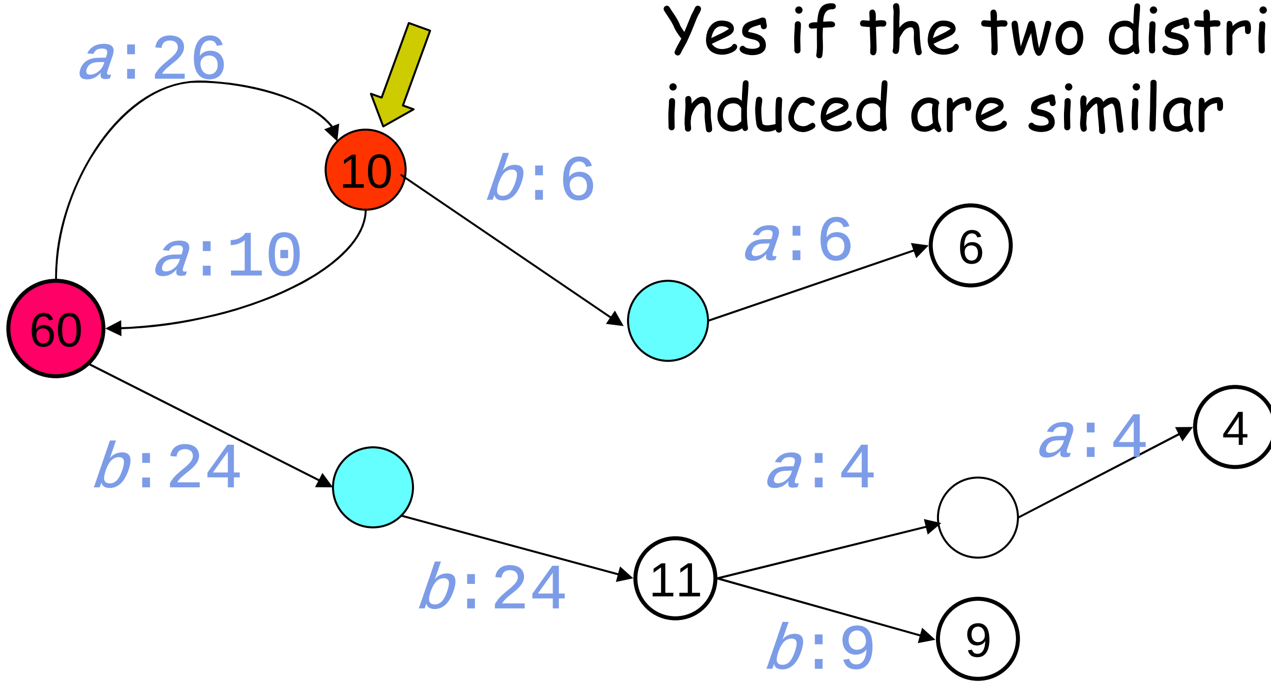
with state

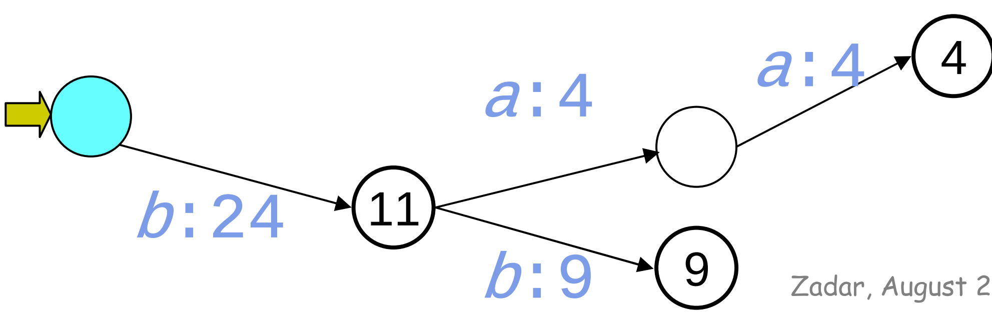




Taking decisions

Yes if the two distributions induced are similar





5 Alergia

Alergia's test

- $\mathcal{D}_1 \approx \mathcal{D}_2$ if $\forall x\, Pr_{\mathcal{D}1}(x) \approx Pr_{\mathcal{D}2}(x)$
- Easier to test:
  - $Pr_{\mathcal{D}1}(\lambda)=Pr_{\mathcal{D}2}(\lambda)$
  - $\forall a\in\Sigma\ Pr_{\mathcal{D}1}(a\Sigma^*)=Pr_{\mathcal{D}2}(a\Sigma^*)$
- And do this recursively!
- Of course, do it on frequencies

Hoeffding bounds

$$\gamma \leftarrow \left| \frac{f_1}{n_1} - \frac{f_2}{n_2} \right|$$

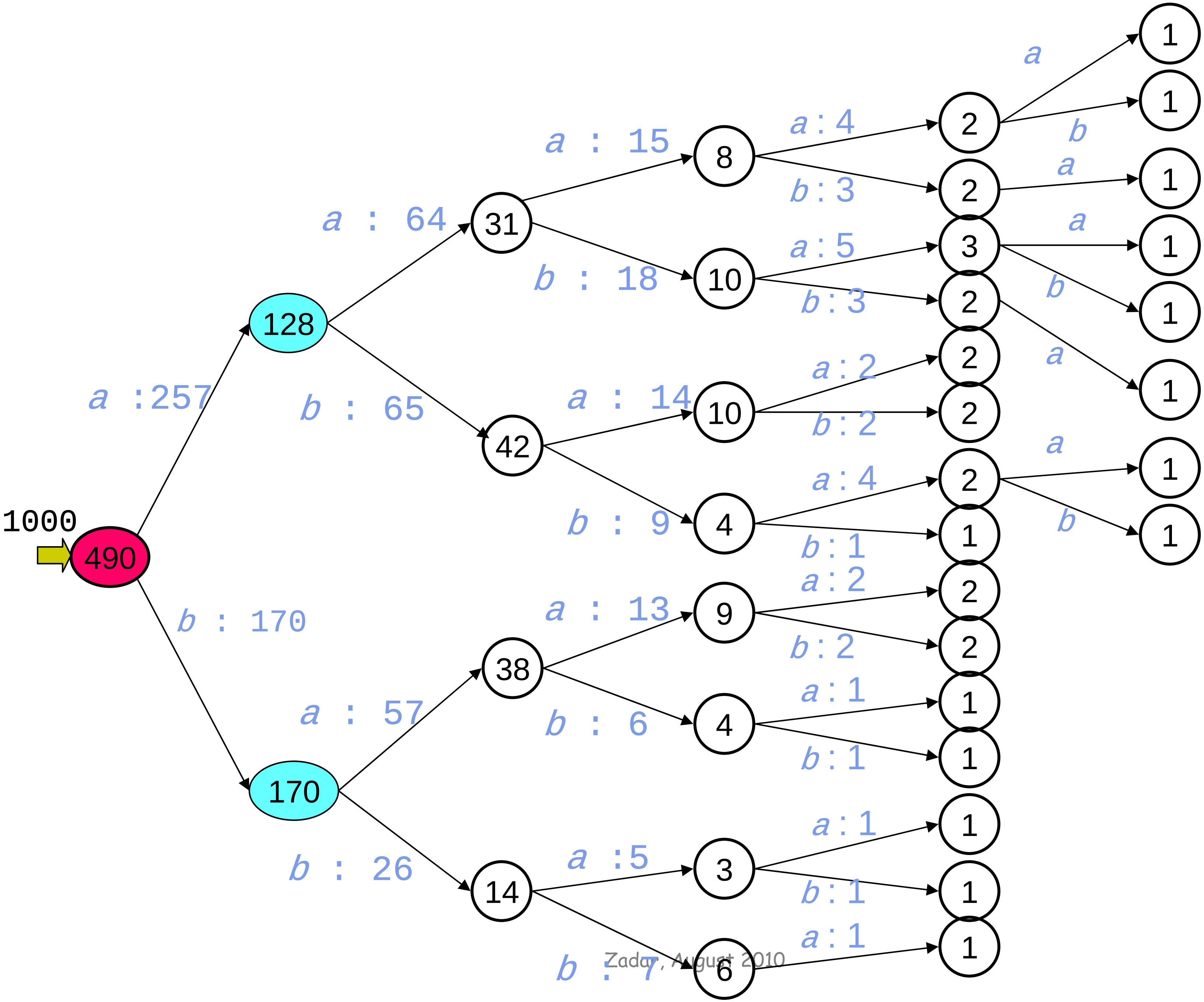
$$\gamma < \left(\sqrt{\frac{1}{n_1}} + \sqrt{\frac{1}{n_2}} \right) . \sqrt{\frac{1}{2} \ln \frac{2}{\alpha}}$$

γ indicates if the relative frequencies $\frac{f_1}{n_1}$ and $\frac{f_2}{n_2}$ are sufficiently close

A run of Alergia
Our learning multisample

S={λ(490), *a*(128), *b*(170), *aa*(31), *ab*(42), *ba*(38), *bb*(14), *aaa*(8), *aab*(10), *aba*(10), *abb*(4), *baa*(9), *bab*(4), *bba*(3), *bbb*(6), *aaaa*(2), *aaab*(2), *aaba*(3), *aabb*(2), *abaa*(2), *abab*(2), *abba*(2), *abbb*(1), *baaa*(2), *baab*(2), *baba*(1), *babb*(1), *bbaa*(1), *bbab*(1), *bbba*(1), *aaaaa*(1), *aaaab*(1), *aaaba*(1), *aabaa*(1), *aabab*(1), *aabba*(1), *abbaa*(1), *abbab*(1)}

- Parameter α is arbitrarily set to 0.05. We choose 30 as a value for threshold t_0.
- Note that for the blue state who have a frequency less than the threshold, a special merging operation takes place


1000
490
a :257
b : 170
128
170
a : 64
b : 65
a : 57
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b : 18
a : 14
b : 9
a : 13
b : 6
a :5
b : 7
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b : 2
a : 4
b : 1
a : 2
b : 2
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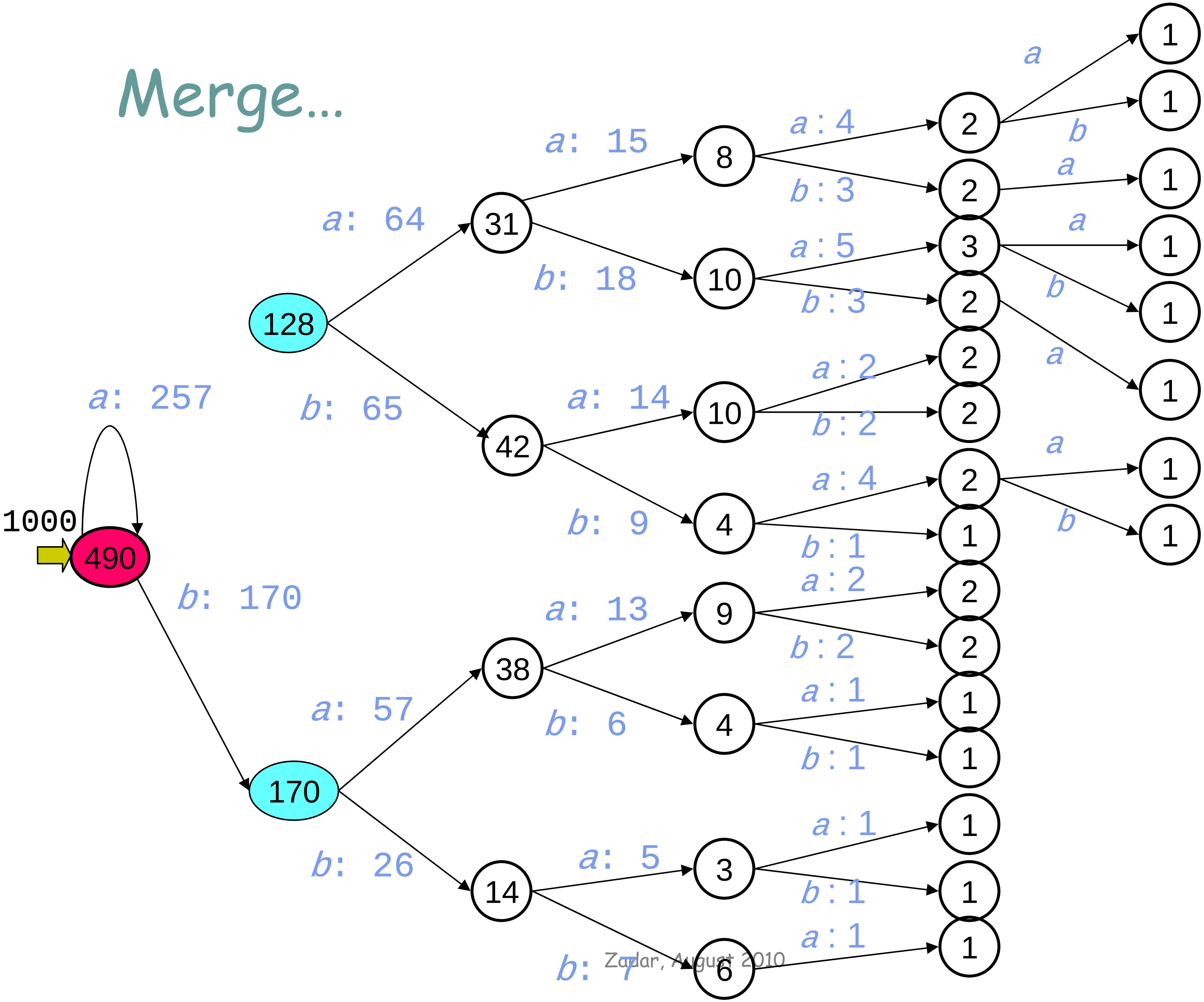
1

Can we merge λ and *a*?

- Compare λ and *a*, *a*Σ* and *aa*Σ*, *b*Σ* and *ab*Σ*
- 490/1000 with 128/257 ,
- 257/1000 with 64/257 ,
- 253/1000 with 65/257 , ...

- All tests return true

Merge…



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b: 170
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170
a: 64
b: 65
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b: 26
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b: 18
a: 14
b: 9
a: 13
b: 6
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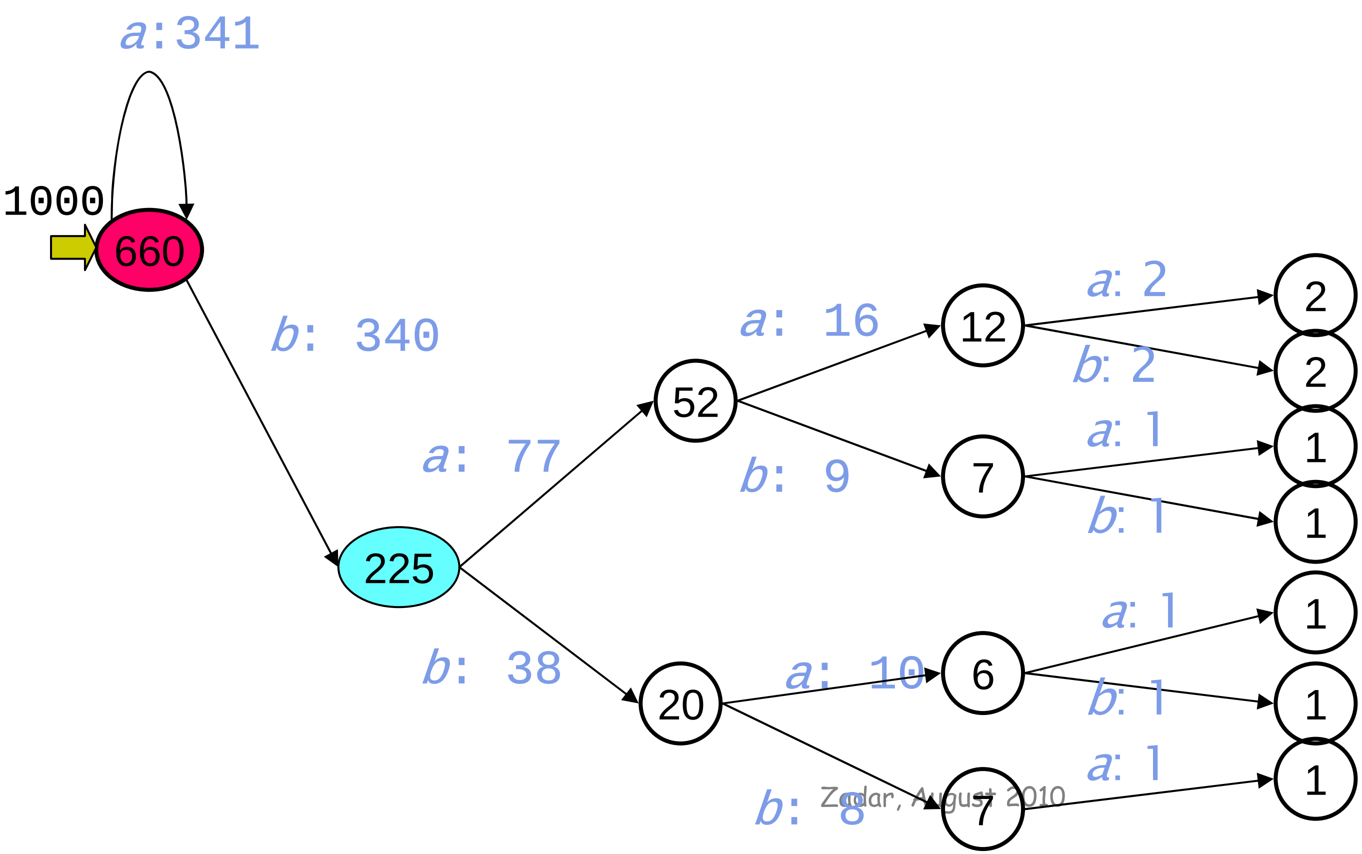
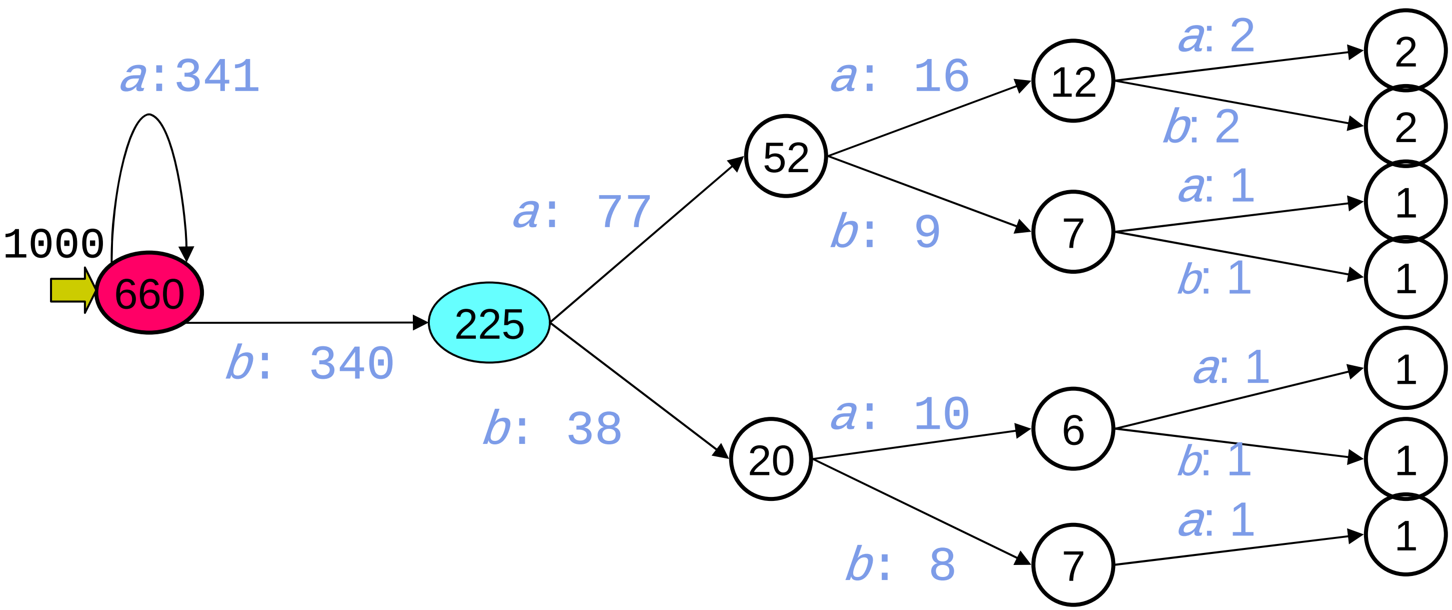
And fold


a:341
1000
660
b: 340
225
a: 77
b: 38
52
20
a: 16
b: 9
a: 10
b: 8
12
7
6
7
a: 2
b: 2
a: 1
b: 1
a: 1
b: 1
a: 1
2
2
1
1
1
1
1

Next merge ?
λ with b ?

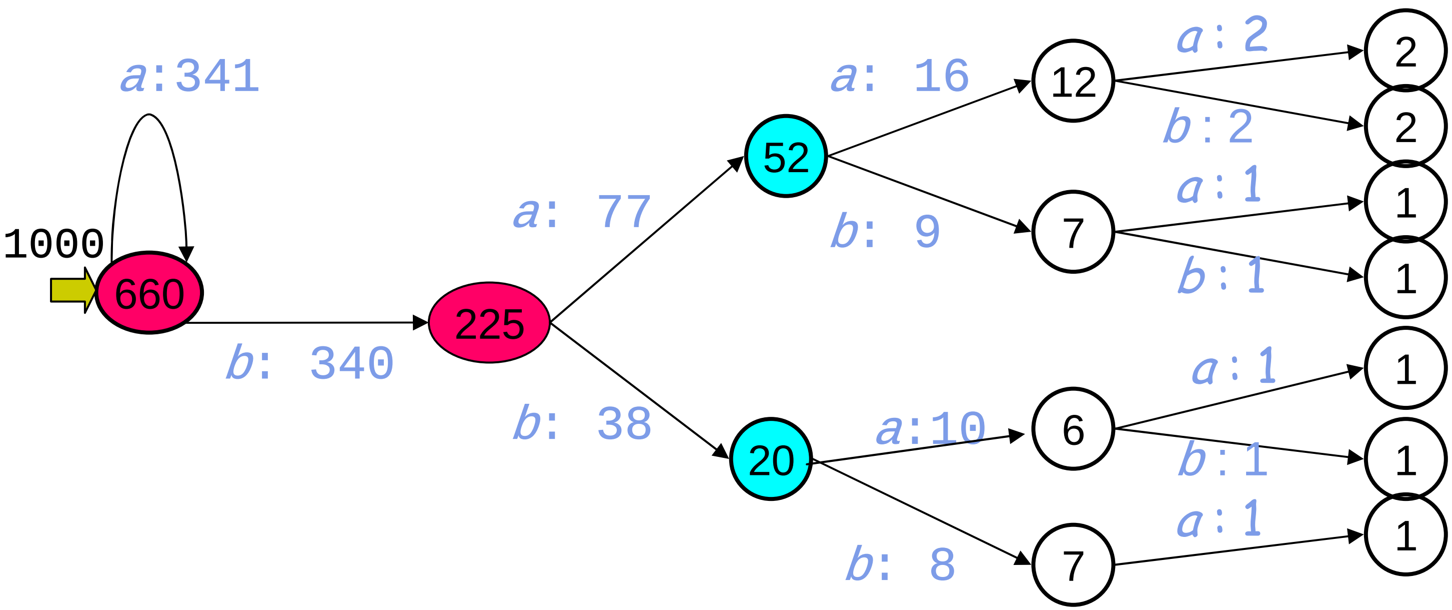


Can we merge λ and *b*?

- Compare λ and *b*, *a*Σ* and *ba*Σ*, *b*Σ* and *bb*Σ*
- 660/1341 and 225/340 are different (giving γ= 0.162)
- On the other hand

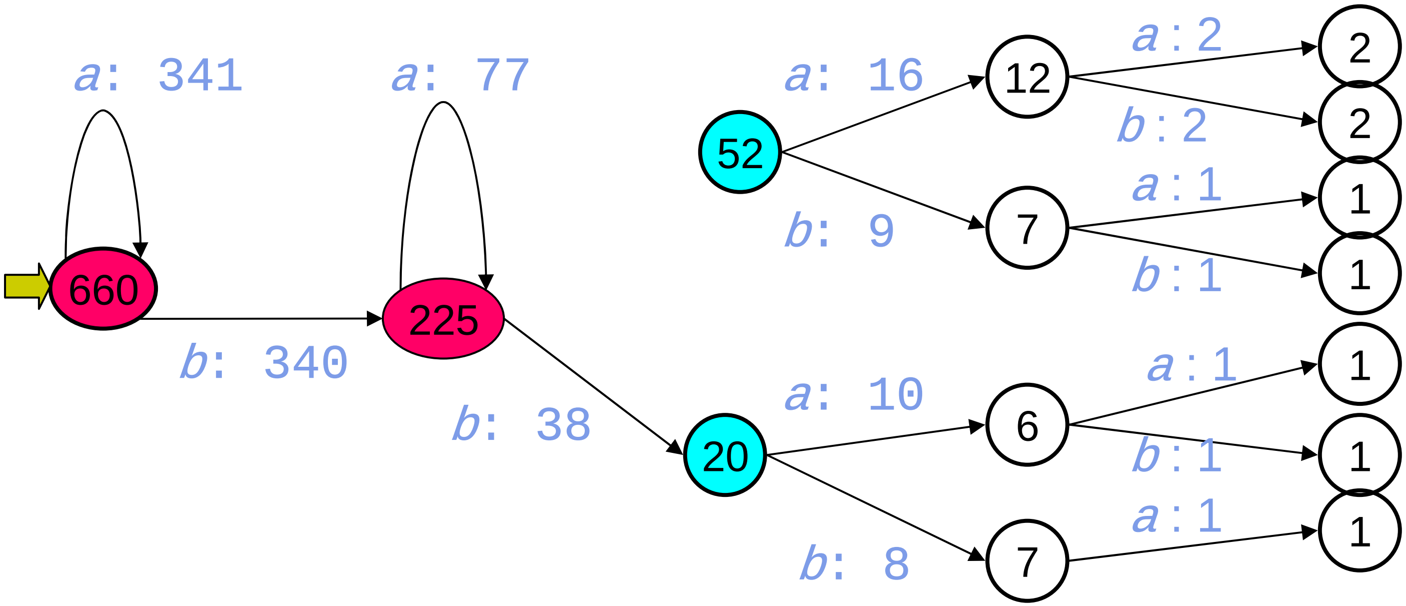
$$\left(\sqrt{\frac{1}{n_1}}+\sqrt{\frac{1}{n_2}}\right)\cdot\sqrt{\frac{1}{2}\ln\frac{2}{\alpha}}=0.111$$

Promotion



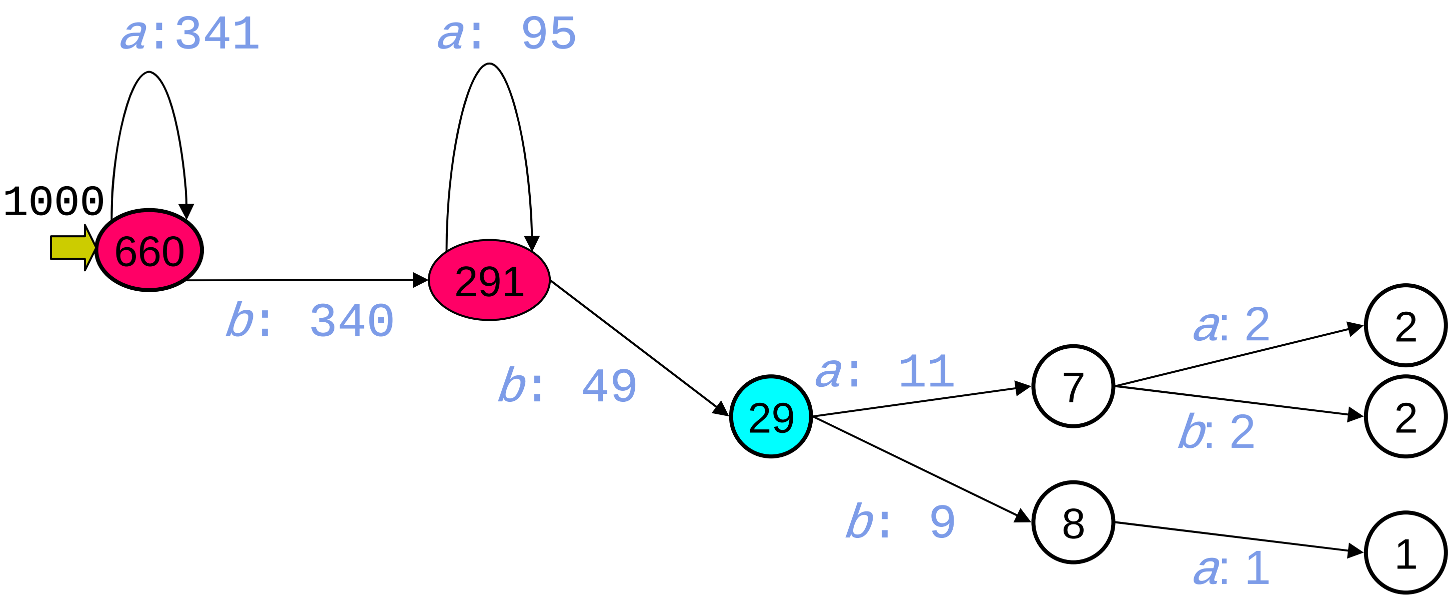
a:341
1000
660
b: 340
225
a: 77
b: 38
52
20
a: 16
b: 9
a:10
b: 8
12
7
6
7
a:2
b:2
a:1
b:1
a:1
b:1
a:1
2
2
1
1
1
1
1

Merge

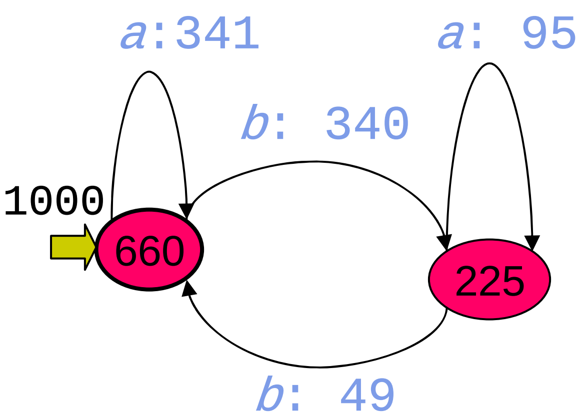


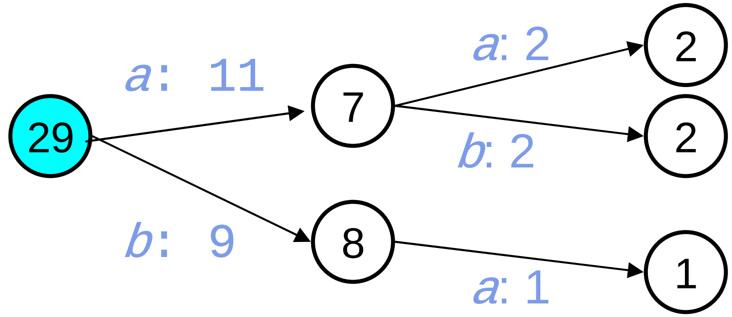
a: 341
a: 77
660
225
b: 340
b: 38
a: 16
b: 9
52
20
a: 10
b: 8
12
7
6
7
a:2
b:2
a:1
b:1
a:1
b:1
a:1
2
2
1
1
1
1
1

And fold

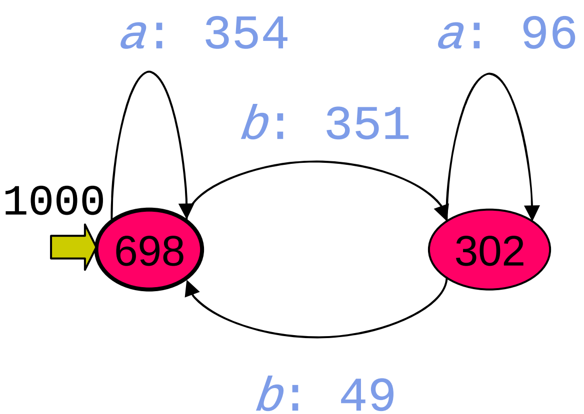


Merge

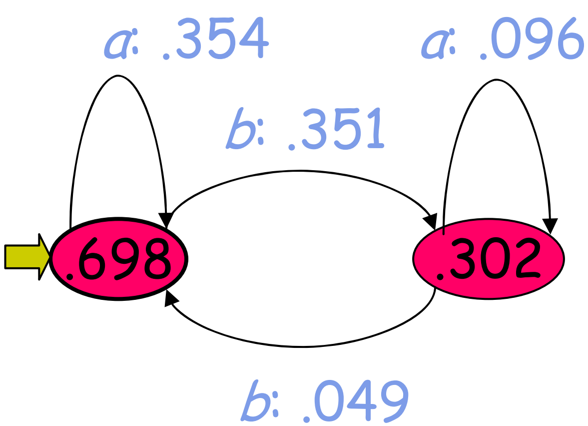




And fold



As a PFA



Conclusion and logic

- Alergia builds a DFFA in polynomial time
- Alergia can identify DPFA in the limit with probability 1
- No good definition of Alergia's properties

6 DSAI and MDI

Why not change the criterion?

Criterion for DSAI

- Using a distinguishable string
- Use norm L_∞
- Two distributions are different if there is a string with a very different probability
- Such a string is called μ-distinguishable
- Question becomes:

Is there a string x such that

$$|Pr_{A,q}(x)-Pr_{A,q'}(x)|>\mu$$

(much more to DSAI)

- D. Ron, Y. Singer, and N. Tishby. On the learnability and usage of acyclic probabilistic finite automata. In *Proceedings of Colt 1995*, pages 31–40, 1995.
- PAC learnability results, in the case where targets are acyclic graphs

Criterion for MDI

- MDL inspired heuristic
- Criterion is: does the reduction of the size of the automaton compensate for the increase in preplexity?
- F. Thollard, P. Dupont, and C. de la Higuera. Probabilistic Dfa inference using Kullback-Leibler divergence and minimality. In *Proceedings of the 17th International Conference on Machine Learning*, pages 975-982. Morgan Kaufmann, San Francisco, CA, 2000

7 Conclusion and open questions

- A good candidate to learn NFA is DEES
- Never has been a challenge, so state of the art is still unclear
- Lots of room for improvement towards probabilistic transducers and probabilistic context-free grammars

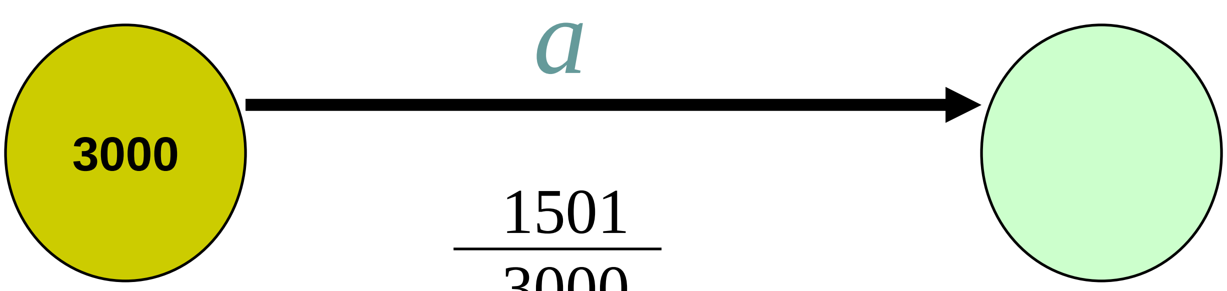
Appendix

Stern Brocot trees

Identification of probabilities

If we were able to discover the structure, how do we identify the probabilities?

- By estimation: the edge is used 1501 times out of 3000 passages through the state :

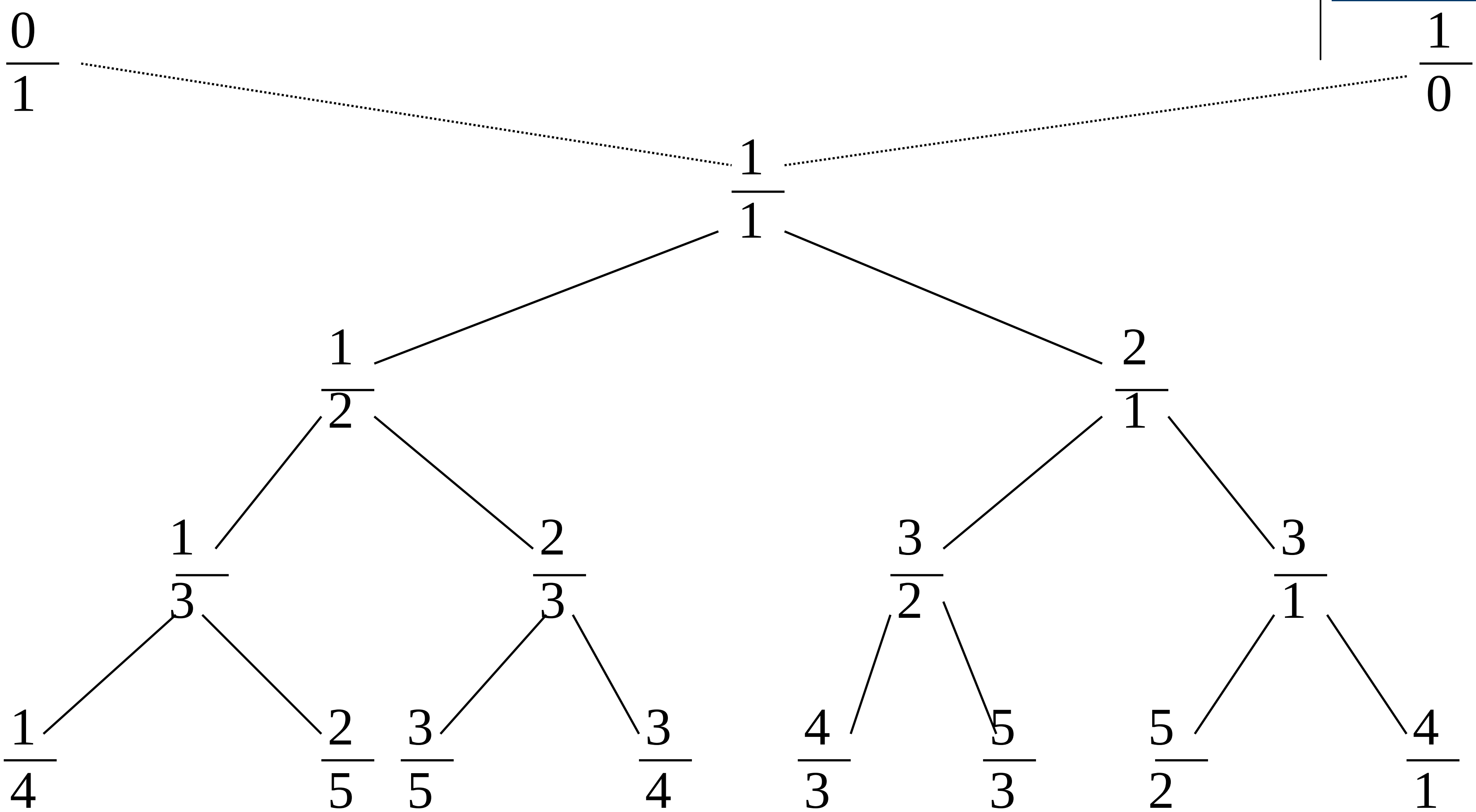


$$\frac{1501}{3000}$$

Stern-Brocot trees: (Stern 1858, Brocot 1860)

Can be constructed from two simple adjacent fractions by the «mean» operation

$$\frac{a}{b} \; m \; \frac{c}{d} = \frac{a+c}{b+d}$$


0/1
1/0
1/1
1/2
2/1
1/3
2/3
3/2
3/1
1/4
2/5
3/5
3/4
4/3
5/3
5/2
4/1

Idea:

- Instead of returning $c(x)/n$, search the Stern-Brocot tree to find a good simple approximation of this value.

Iterated Logarithm:

With probability 1, for a co-finite number of values of *n* we have:

$$\left| \frac{c(x)}{n} - \frac{a}{b} \right| < \sqrt{\frac{\lambda \log \log n}{n}}$$

$$\forall \lambda > 1$$